

11.1 Approx. Functions w/ Polynomials (part 2)

Taylor series of $f(x)$ near $x=a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

if we chop it off after k , we get the k -th order Taylor Polynomial
for example, $k=2$,

$$P_2 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 \approx f(x) \quad \text{but NOT exactly equal}$$

the terms we throw away ($k \geq 3$) form the Remainder

here, the remainder ($k \geq 3$)

remainder $\rightarrow$

$$R_2 = \frac{f'''(a)}{3!} (x-a)^3 + \frac{f^{(4)}(a)}{4!} (x-a)^4 + \dots$$

after 2nd
order

itself an infinite series
which converges to one term

the remainder converges to

$$R_k = \frac{f^{(k+1)}(c)}{(k+1)!} (x-a)^{k+1}$$

where $a < c < x$

Taylor's Remainder Theorem

so, in general, approximation of $f(x)$ using k^{th} order Taylor Polynomial

$$f(x) = \underbrace{P_k(x)}_{\substack{k^{\text{th}} \text{ order} \\ \text{Taylor} \\ \text{Polynomial}}} + \underbrace{R_k(x)}_{\text{remainder}}$$

c is usually not easy to find, so instead of finding it, we usually try to put a bound on $|R_k|$ and use it to estimate error

the absolute error is

$$\left| \underset{\text{true}}{f(x)} - \underset{\substack{k\text{th order} \\ \text{Taylor polynomial}}}{P_k(x)} \right| = |R_k| = \left| \frac{f^{(k+1)}(c)}{(k+1)!} (x-a)^{k+1} \right|$$

we don't find c but instead we find the max $|f^{(k+1)}(x)|$
and call it M

then the |error| is no more than $\left| \frac{M}{(k+1)!} (x-a)^{k+1} \right|$

example

$$f(x) = e^{-2x}$$

$$a = 0$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

approx using P_2 and estimate the error.

$$f(x) = e^{-2x}$$

$$f(a) = f(0) = e^0 = 1 = (-2)^0$$

$$f'(x) = -2e^{-2x}$$

$$f'(a) = f'(0) = -2e^0 = -2 = (-2)^1$$

$$f''(x) = 4e^{-2x}$$

$$f''(0) = 4 = (-2)^2$$

$$f'''(x) = -8e^{-2x}$$

$$f'''(0) = -8 = (-2)^3$$

$\vdots$

$$f^{(k)}(0) = (-2)^k$$

Taylor series

$$e^{-2x} = 1 - 2x + \frac{(-2)^2}{2!} x^2 + \frac{(-2)^3}{3!} x^3 + \frac{(-2)^4}{4!} x^4 + \dots$$

estimate using $P_2(x)$

$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!} x^2 = 1 - 2x + 2x^2$$

with remainder

$$R_2(x) = \frac{(-2)^3}{3!} x^3 + \frac{(-2)^4}{4!} x^4 + \dots = \frac{f'''(c)}{3!} x^3$$

$$\begin{array}{c} 0 \\ \swarrow \\ a < c < x \end{array}$$

$$f'''(x) = -8e^{-2x}$$

$$\frac{f'''(x)}{3!} x^3 = \frac{-8e^{-2x}}{3!} x^3$$

$$\frac{f'''(c)}{3!} x^3 = -\frac{4}{3} \textcircled{e^{-2c}} x^3$$

$$0 < c < x$$

now let's use $P_2(x)$ to estimate $e^{-0.02}$ (as an example)

$$e^{-2x} \approx P_2(x) = 1 - 2x + 2x^2$$

$$e^{-0.02} \approx P_2(0.01) = 1 - 2(0.01) + 2(0.01)^2 = \frac{4901}{5000} \quad (0.9802)$$

what is the error?

it is exactly equal to $|R_2(c)|$ but we don't know c

$$0 < c < 0.01$$

$\nearrow$ a $\nwarrow$ x

can we at least bound e^{-2c} ?

e^{-2x} is a monotonic decreasing function, so on $0 < c < 0.01$
 the maximum of $|e^{-2c}|$ is at the left end ($x=0$)

$$|e^{-2c}| \leq |e^{-2(0)}| \leq 1 \quad \text{the max } |e^{-2c}| \text{ can be is } 1$$

$$\begin{aligned} \text{so, } |R_2(x)| &\leq \left| \frac{f'''(c)}{3!} (0.01)^3 \right| \\ &\leq \left| -\frac{4}{3} \underbrace{\frac{f'''(c)}{e^{-2c}}}_{1} (0.01)^3 \right| = \frac{4}{3000000} \end{aligned}$$

so, our approx. of $e^{-0.02} = \frac{4501}{5000}$ is no more than $\frac{4}{3000000}$ off

$$\frac{4501}{5000} - \frac{4}{3000000} \leq e^{-0.02} \leq \frac{4501}{5000} + \frac{4}{3000000}$$



$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!} x^2 + \frac{(-2)^3}{3!} x^3 + \frac{(-2)^4}{4!} x^4 + \frac{(-2)^5}{5!} x^5 + \dots$$

as example, to estimate, for example, e^{-1} using P_3

$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!} x^2 + \frac{(-2)^3}{3!} x^3$$

$$e^{-1} = e^{-2(\frac{1}{2})} \approx 1 - 2(\frac{1}{2}) + \frac{(-2)^2}{2!} (\frac{1}{2})^2 + \frac{(-2)^3}{3!} (\frac{1}{2})^3$$

$$\approx 1 - 1 + (\frac{1}{2}) - \frac{4}{3} (\frac{1}{8})$$

$$\approx \frac{1}{2} - \frac{1}{6} \approx \frac{1}{3}$$

$$|\text{error}| = R_3(x) = \frac{f^{(4)}(a)}{4!} x^4 + \frac{f^{(5)}(a)}{5!} x^5 + \dots$$

$$= \left| \frac{f^{(4)}(c)}{4!} x^4 \right|$$

bound $|f^{(4)}(c)|$ instead
of finding c