11.2 Properties of Power Series (part 1)

power series : $\sum_{k=0}^{\infty} c_k (x-a)^k$ $\quad$ a: center $\quad$ c_k : coefficients of k^{th}-order term

convergence in general depends on x

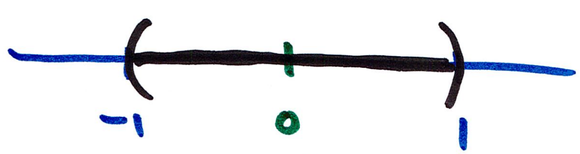
the interval on which the power series converges → interval of convergence
(already saw this in Ratio Test section)

for example, $\sum_{k=0}^{\infty} x^k$ $\qquad c_k = 1, \quad a = 0$

converges if $|x| < 1$

$-1 < x < 1$

Sketch:



this is the radius of convergence (center to one end) example $$\sum_{k=1}^{\infty} \underbrace{\frac{(-1)^k k}{4^k}}_{c_k} \underbrace{(x+3)^k}_{(x-a)^k}$$

so $a = -3$ (center of interval of convergence)

find interval of convergence

Ratio Test is good for this

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}(k+1)}{4^{k+1}}(x+3)^{k+1}}{\frac{(-1)^k(k)}{4^k}(x+3)^k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)}{4^{k+1}} \cdot \frac{4^k}{(-1)^k(k)}(x+3) \right|$$

$$= \lim_{k\to\infty} \left| (-1)\underbrace{\frac{(k+1)}{k}}_{1} \cdot \frac{1}{4} \cdot (x+3) \right| = \left| \frac{x+3}{4} \right| < 1$$

ratio < 1 converges

$|x+3| < 4$

$-4 < x+3 < 4$

$-7 < x < 1$

this is NOT it! At ends the ratio is 1 (inconclusive)

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (x+3)^k$$

at $x=-7$ $\quad \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (-4^k) = \cancel{\sum_{k=1}^{\infty} (-1)^k}\, k$ $\quad$ diverges at $x=-7$

$$= \sum_{k=1}^{\infty} k$$

at $x=1$ $\quad \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (4^k) = \sum_{k=1}^{\infty} (-1)^k k$ $\quad$ diverges at $x=1$

so interval of convergence is $-7 < x < 1$ or $(-7, 1)$

radius of conv.
$R = 4$

example $\displaystyle\sum_{k=1}^{\infty} (kx)^k = \sum_{k=1}^{\infty} k^k \cdot x^k = \sum_{k=1}^{\infty} \underbrace{k^k}_{c_k} \cdot (x-0)^k$ $\quad a=0$

Ratio Test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^{k+1} \cdot x^{k+1}}{k^k \cdot x^k} \right| < 1$$

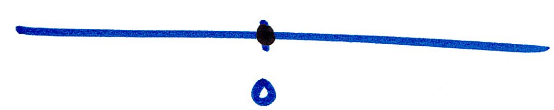
$$= \lim_{k\to\infty} \left| \underbrace{\left(\frac{k+1}{k}\right)^k}_{\substack{(1+\frac{1}{k})^k \\ e}} \cdot \underbrace{(k+1)}_{\infty} \cdot x \right| = \infty \quad \text{unless } x=0$$

ratio < 1 only if $x = 0$

the only value of x for which the series converges is $x=0$

interval of convergence : $x=0$

radius of conv: $R=0$



series when $x=0$ $\displaystyle\sum_{k=1}^{\infty} k^k \cdot (0)^k = \sum_{k=1}^{\infty} 0 = 0+0+0+0+0+0+\ldots$

Summation notation is useful when finding intervals of convergence

you may need to find it yourself

<u>example</u> $x - \frac{x^3}{4} + \frac{x^5}{9} - \frac{x^7}{16} + \cdots$ in summation notation?

patterns? numerator: x to powers of odd numbers, alternating

denominator: something squared

$$= \frac{x^1}{1^2} - \frac{x^3}{2^2} + \frac{x^5}{3^2} - \frac{x^7}{4^2} + \frac{x^9}{5^2} - \frac{x^{11}}{6^2} + \cdots$$

$k=1$ $\quad k=2$ $\quad k=3$ $\quad k=4$ $\quad k=5$ $\quad k=6$

decide a starting k

here, $k=1$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{k^2}$$

power series as functions

we know $\sum_{k=0}^{\infty} ar^k = a + ar^1 + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ if $|r|<1$

treat it as a function: $r = x$, $a = 1$

$$\frac{a}{1-r} = \boxed{\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + \underbrace{x + x^2 + x^3 + x^4}_{} + \cdots \quad \text{if } |x|<1}$$

Taylor series of $\frac{1}{1-x}$

(without taking derivatives!)

we can modify it to find Taylor series of similar functions

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

now replace x in

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

w/ $(-x)$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

converges if $|-x| < 1$

$|x| < 1$

$$\frac{x^2}{1+x^2} \quad \text{Taylor series?}$$

$$= x^2 \cdot \frac{1}{1-(-x^2)} = x^2 \sum_{k=0}^{\infty} (-x^2)^k = x^2 \sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-1)^k x^{2k+2}$$

rewrite $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

change x to $-x^2$

$$|-x^2| < 1$$

$$x^2 < 1$$

$$-1 < x < 1$$

$$= x^2(1 - x^2 + x^4 - x^6 + x^8 - \cdots)$$

$$= x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots.$$

the "1" in $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ is important

$\frac{1}{2-x}$ must rearrange so the denom. starts w/ 1

$$= \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2} \cdot \underbrace{\frac{1}{1-\left(\frac{x}{2}\right)}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^k}{2^k} = \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}$$

reuse

$\Leftarrow \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$|x| < 1$

interval of conv: we changed x to $\frac{x}{2}$

$$|"x"| < 1 \implies \left|\frac{x}{2}\right| < 1$$

$$|x| < 2$$

$$-2 < x < 2$$