

11.4 Working with Taylor Series

$$\text{Taylor series: } f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

any differentiable $f(x)$ near $x=a$ can be expressed as a Taylor series

if $a=0$, Taylor $\rightarrow$ Maclaurin

usually we work with "model" Maclaurin series

Common Maclaurin Series $a=0$ only

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^k + \dots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^k x^k + \dots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^k x^{2k}}{(2k)!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{k+1} x^k}{k} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \leq 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^k}{k} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \leq x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^k x^{2k+1}}{2k+1} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \leq 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2k+1}}{(2k+1)!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2k}}{(2k)!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

example

$$f(x) = \frac{9 \tan^{-1} x - 9x + 3x^3}{5x^5}$$

Maclaurin series?

build from Table of "model" series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

$$9 \tan^{-1} x = 9x - 3x^3 + \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \dots$$

$$9 \tan^{-1} x - 9x = -\frac{9}{3}x^3 + \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \dots$$

$$9 \tan^{-1} x - 9x + 3x^3 = \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \frac{9}{11}x^{11} + \dots$$

$$\frac{9 \tan^{-1} x - 9x + 3x^3}{5x^5} = \frac{9}{5 \cdot 5} - \frac{9}{5 \cdot 7}x^2 + \frac{9}{5 \cdot 9}x^4 - \frac{9}{5 \cdot 11}x^6 + \frac{9}{5 \cdot 13}x^8 - \dots$$

$$= \frac{9}{5} \left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \frac{x^6}{11} + \frac{x^8}{13} - \frac{x^{10}}{15} + \dots \right)$$

$k=0$ $k=1$ $k=2$ $k=3$

choose to
start at
 $k=0$

$$= \boxed{\frac{9}{5} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k+5}}$$

near $x=0$, this behaves like $f(x)$

how to use that?

$$\lim_{x \rightarrow 0} \frac{9 \tan^{-1} x - 9x + 3x^3}{5x^5}$$

$\hookrightarrow$
implies x is near 0
so Maclaurin series
can be used in place
of $f(x)$

usually we use l'Hospital's Rule

$$\rightarrow \lim_{x \rightarrow 0} \frac{9}{5} \left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \dots \right) = \frac{9}{25}$$

Maclaurin of $f(x)$

another good use

$$\int_0^1 \frac{9 \tan^{-1} x - 9x + 3x^3}{5x^5} dx$$

$\rightarrow$ implies x is near 0

$$= \int_0^1 \frac{9}{5} \left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \frac{x^6}{11} + \dots \right) dx$$

$$= \frac{9}{5} \left(\frac{1}{5}x - \frac{x^3}{3 \cdot 7} + \frac{x^5}{5 \cdot 9} - \frac{x^7}{7 \cdot 11} + \frac{x^9}{9 \cdot 13} - \frac{x^{11}}{11 \cdot 15} + \dots \right) \Big|_0^1$$

$$= \frac{9}{5} \left(\frac{1}{5} - \frac{1}{21} + \frac{1}{45} - \frac{1}{77} + \frac{1}{117} - \dots \right) \quad \text{let's cut off after } \frac{1}{45}$$

approx. of
the integral

$$\int_0^1 \frac{9 \tan^{-1} x - 9x + 3x^3}{5x^5} dx \approx \frac{9}{5} \left(\frac{1}{5} - \frac{1}{21} + \frac{1}{45} \right) \approx 0.3143$$

what is the error?

one option: Taylor's Remainder Theorem

easier $\swarrow$ Alternating Series Estimation
(because this happens to be
an alt. series)

$$|\text{error}| \leq |\text{first term we throw out}|$$

$$\leq \left| \frac{9}{5} \cdot -\frac{1}{77} \right| \approx 0.0234$$

example Estimate $\int_0^{0.1} e^{-x^2} dx$ to within 10^{-8}

near 0 $\rightarrow$ MacLaurin is good

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \dots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots \end{aligned}$$

$$\int_0^{0.1} e^{-x^2} dx = \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots \right) dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!} + \frac{x^{13}}{13 \cdot 6!} - \dots \Bigg|_0^{0.1}$$

$$= 0.1 - \underbrace{\frac{(0.1)^3}{3}}_{3 \times 10^{-4}} + \underbrace{\frac{(0.1)^5}{5 \cdot 2!}}_{10^{-6}} - \underbrace{\frac{(0.1)^7}{7 \cdot 3!}}_{2.38 \times 10^{-9}} + \frac{(0.1)^9}{9 \cdot 4!} - \dots$$

↓
first time below 10^{-8}

so sum up to the term before it

alt. series

$|error| \leq | \text{first term we throw out} |$

goal: $|error| \leq 10^{-8}$

$$\approx 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5 \cdot 2!} \approx 0.099068 \approx \int_0^{0.1} e^{-x^2} dx \quad (\text{to within } 10^{-8})$$

$$\int e^{-x} dx = -e^{-x} + c$$

$\int e^{-x^2} dx =$ cannot be expressed using elementary functions
in closed form

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

Alt. Harmonic
converges

$$= S \quad (\text{sum of this series})$$

has a sum because it converges

$$S = ?$$

$$\text{from Table: } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$



alt. harmonic if $x = 1$

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \boxed{\ln(2)}$$

↑
S