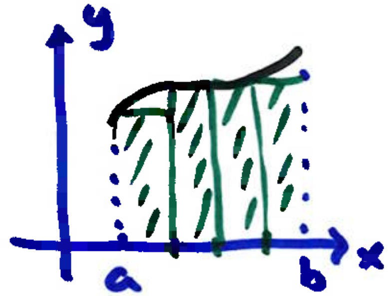
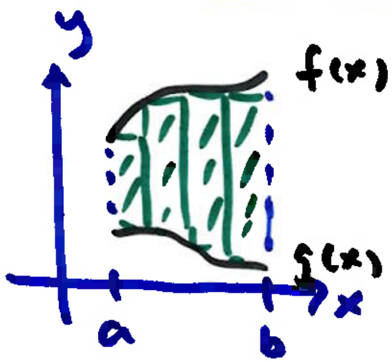
12.3 Areas and Lengths in Polar Coordinates

In Cartesian, $\int_a^b f(x)\,dx$



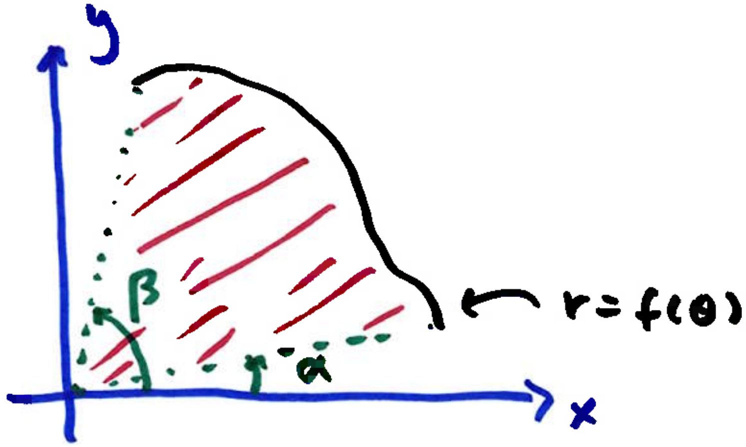
shrink rectangles then sum infinitely-many of them



$$\int_a^b [f(x) - g(x)]\,dx$$

Sum of infinitely-many rectangles between f(x) and g(x)

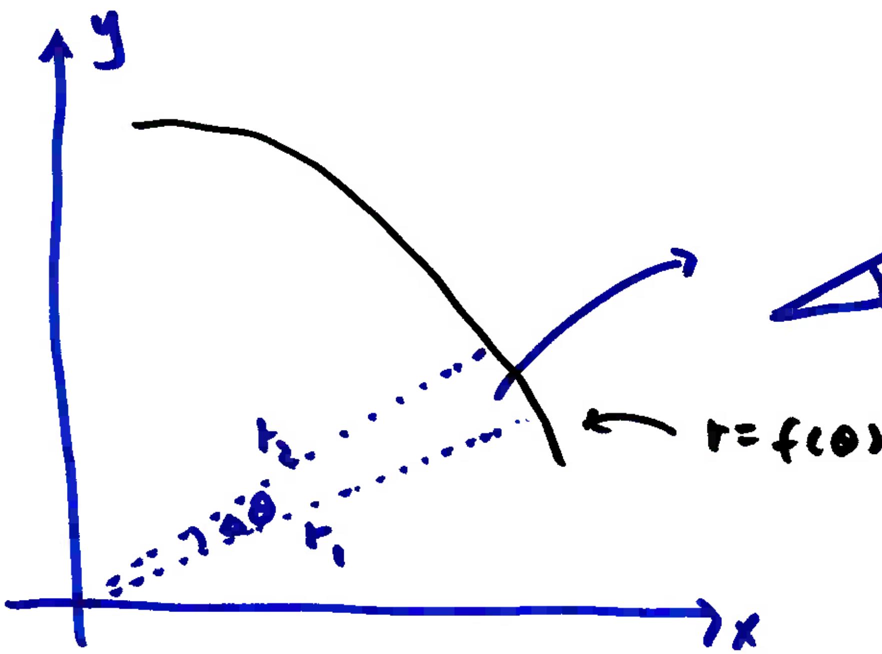
In Polar, same idea, but instead of rectangles, we use thin slices of circles

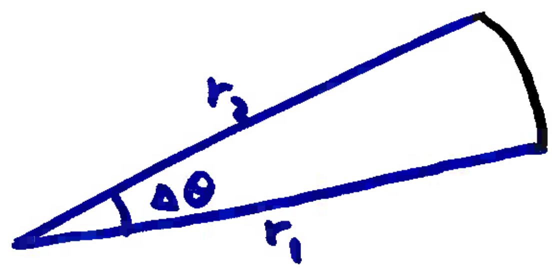


curve in polar equation

area of red region?

divide into thin slices





when $\Delta\theta$ is small, $r_1 \approx r_1 = r$

from geometry, the area of this segment is

$$\boxed{\frac{1}{2} r^2 \Delta\theta}$$

this is the polar equivalent of a rectangle

Sum up these slices from α to β, and shrink $\Delta\theta \to d\theta$

so, the entire region has area

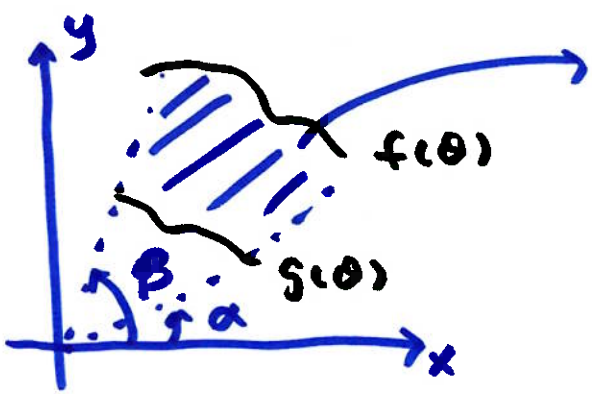
$$\boxed{\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta}$$

function of θ

if $r = f(\theta)$

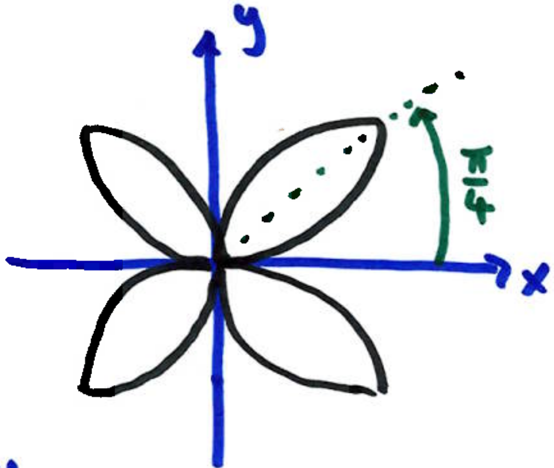
$$\boxed{\int_\alpha^\beta \frac{1}{2} [f(\theta)]^2 \, d\theta}$$

between curves

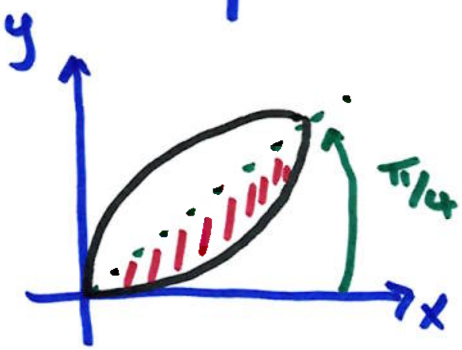


$$\int_\alpha^\beta \left\{ \frac{1}{2} [f(\theta)]^2 - \frac{1}{2} [g(\theta)]^2 \right\} d\theta$$

example Find area of one petal of the rose $r = \sin 2\theta$



find area of any, for simplicity, let's find QI



notice symmetry, so we can find area of half (red portion) then double it

$$\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

$\sin 2\theta$

double

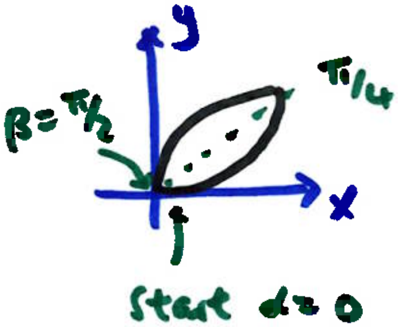
to middle of petal

$$2\int_0^{\pi/4} \frac{1}{2}(\sin 2\theta)^2 \, d\theta = \int_0^{\pi/4} \sin^2 2\theta \, d\theta = \int_0^{\pi/4} \frac{1-\cos(4\theta)}{2} \, d\theta$$

minus

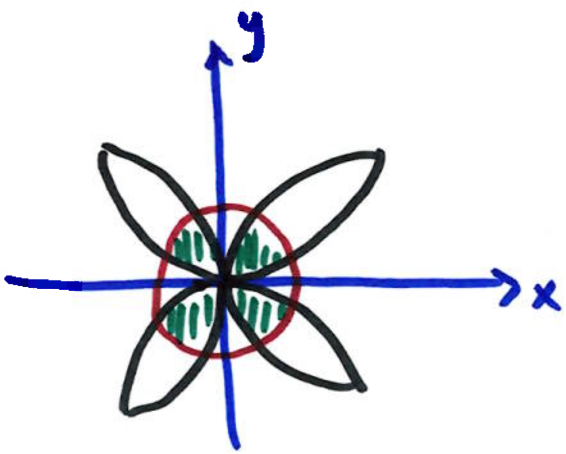
$$= \frac{1}{2}\int_0^{\pi/4} 1-\cos(4\theta) \, d\theta = \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{8}}$$

alternative: ~~no~~ w/o using symmetry

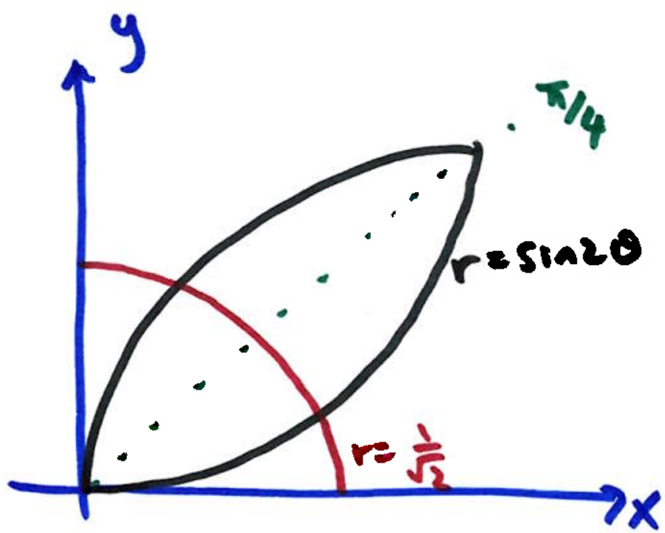


$$\int_0^{\pi/2} \frac{1}{2}(\sin 2\theta)^2 \, d\theta = \cdots = \frac{\pi}{8}$$

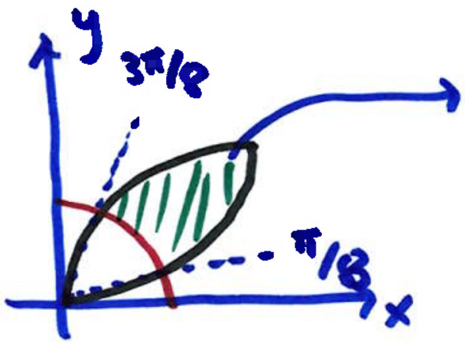
<u>example</u> Area bounded by ~~inside~~ $\underbrace{r = \frac{1}{\sqrt{2}}}_{\text{circle radius } \frac{1}{\sqrt{2}}}$ and ~~out~~ $\underbrace{r = \sin 2\theta}_{\text{rose we looked at}}$ closer to origin



QI:



one approach: find area of rose petal outside circle, then subtract from area of one petal



$$\int_{\pi/8}^{3\pi/8} \left[\underbrace{\frac{1}{2}(\sin 2\theta)^2 \cancel{d\theta}}_{\substack{\text{outside} \\ \text{(rose)}}} - \underbrace{\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2}_{\substack{\text{inside} \\ \text{(circle)}}} \right] d\theta$$

intersection: $r = \frac{1}{\sqrt{2}}$
and
$r = \sin 2\theta$
are equal

Solve: $\frac{1}{\sqrt{2}} = \sin 2\theta$

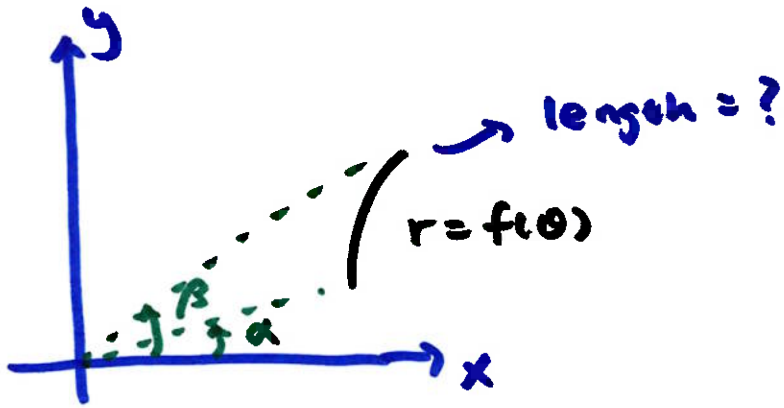
$\vdots$

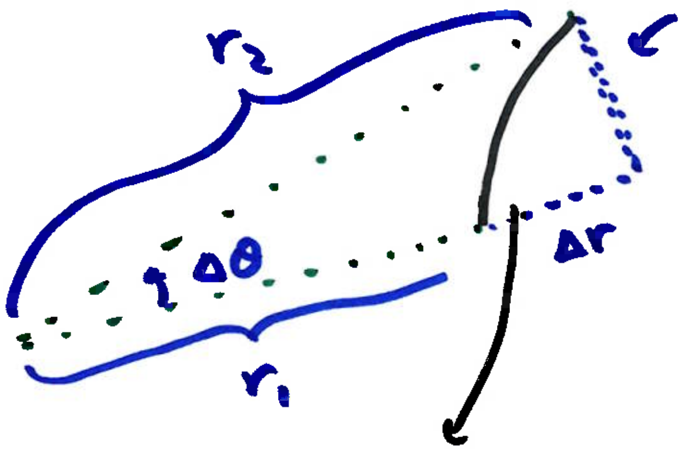
$\theta = \frac{\pi}{8}$

then, subtract from rose petal area

last example

Arc length :

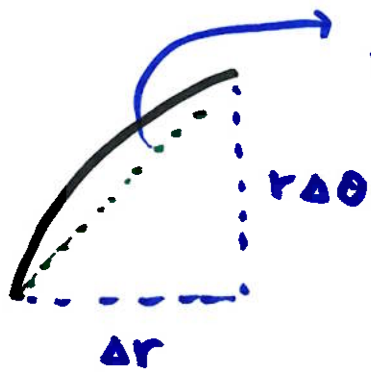




circular arc

length $= r\Delta\theta$

when $\Delta\theta$ is small, $r_1 \approx r_2 = r$



$$\sqrt{(\Delta r)^2 + (r\Delta\theta)^2} \approx \text{length of black curve}$$

$$\hookrightarrow \sqrt{(\Delta\theta)^2 \left[\frac{(\Delta r)^2}{(\Delta\theta)^2} + r^2 \right]}$$

$$= \sqrt{\left[\left(\frac{\Delta r}{\Delta\theta}\right)^2 + r^2\right](\Delta\theta)^2} \quad = \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}\ \Delta\theta$$

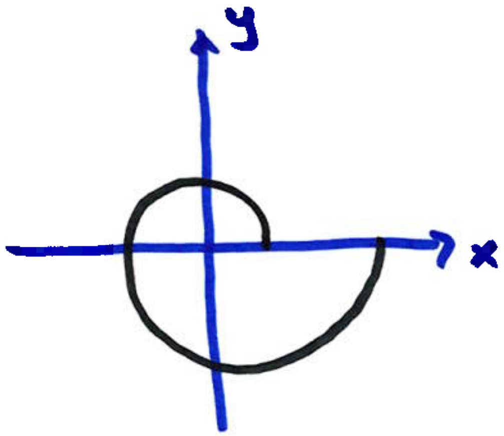
Shrink : $\frac{\Delta r}{\Delta\theta} \to \frac{dr}{d\theta}$

$\Delta\theta \to d\theta$

accumulate from $\theta = \alpha$ to $\theta = \beta$ by integration

$$\boxed{\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta}$$

__example__ Length of $r = e^{\theta}$ $\quad 0 \le \theta \le 2\pi$



logarithmic spiral

length: $\displaystyle\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r = e^{\theta}$

$\dfrac{dr}{d\theta} = e^{\theta}$

$$\int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$(e^{2\theta})^{1/2}$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{e^{2\theta}}\, d\theta = \int_0^{2\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2} \int_0^{2\pi} e^{\theta}\, d\theta$$

$$= \sqrt{2}\left(e^{\theta}\right)\Big|_0^{2\pi} = \boxed{\sqrt{2}\left(e^{2\pi} - 1\right)}$$