

13.4 The Cross Product

Another way vectors multiply

dot product: $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$



$$\vec{u} = \langle a, b, c \rangle \quad \vec{v} = \langle d, e, f \rangle$$

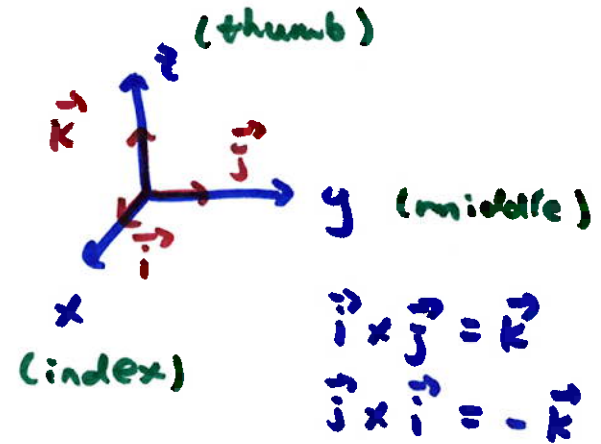
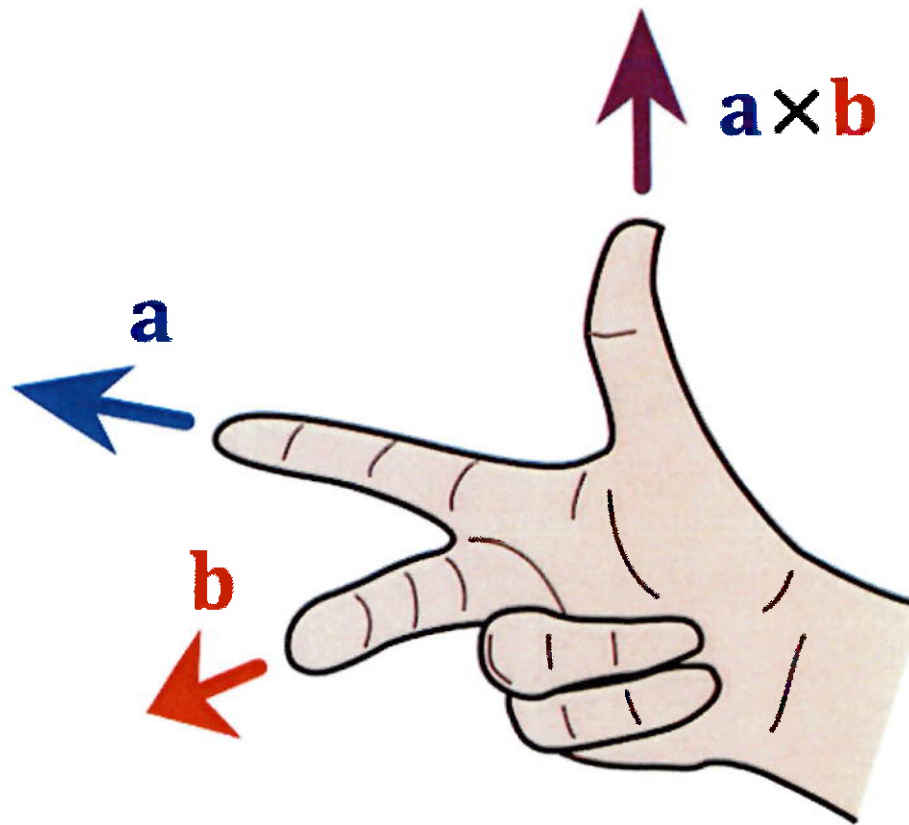
$$\vec{u} \cdot \vec{v} = ad + be + cf \quad \text{Scalar}$$

the cross product of $\vec{u}, \vec{v}$ is $\vec{u} \times \vec{v}$ (or $\vec{v} \times \vec{u}$)

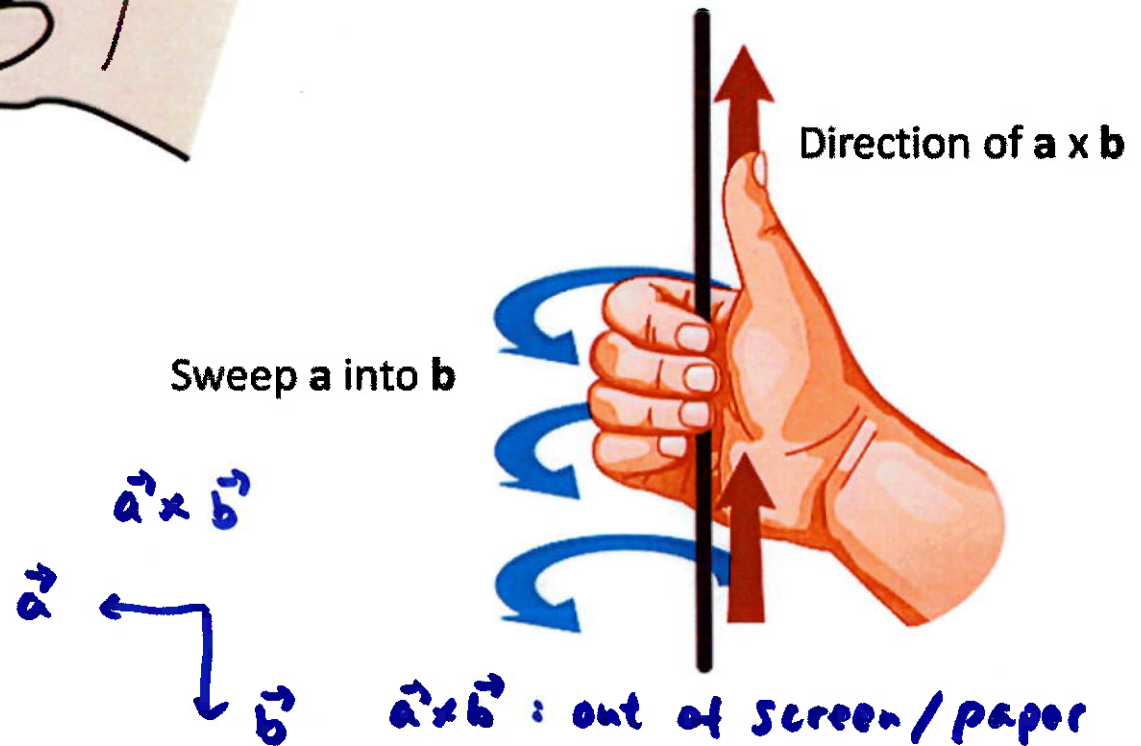
the result is a vector

$$\text{magnitude: } |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$$

direction: by the right-hand rule



CURL RIGHT HAND RULE

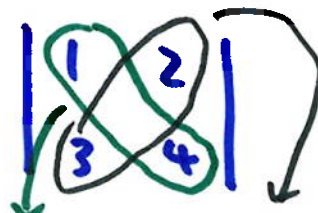


$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u}) \quad \text{In cross product, order matters}$$

how to compute $\vec{u} \times \vec{v}$?

one way; use $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$ to find magnitude
then right-hand rule for direction

the other (algebraic) way: the determinant of a special matrix

2x2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ determinant is 
 $= (1)(4) - (2)(3)$
 $= 4 - 6 = -2$

$$\begin{vmatrix} -3 & 1 \\ 7 & -4 \end{vmatrix} = (-3)(-4) - (1)(7) = 12 - 7 = 5$$

we need determinant of a special 3x3 matrix for cross product

$$\vec{u} = \langle 2, 1, 2 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$$

find $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

first row: $\vec{i}, \vec{j}, \vec{k}$

2nd row: first vector
in cross product

3rd row: the second vector

move along first row

$$= \vec{i} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix}$$

$\vec{j}$ gets negative
 determinant of leftover after removing row and column of $\vec{i}$
 determinant of leftover after removing row, column $\vec{j}$ is in

$$= \vec{i}(1) - \vec{j}(-8) + \vec{k}(-5) = \vec{i} + 8\vec{j} - 5\vec{k} = \langle 1, 8, -5 \rangle$$

$$\vec{u} = \langle 2, 1, 2 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$$

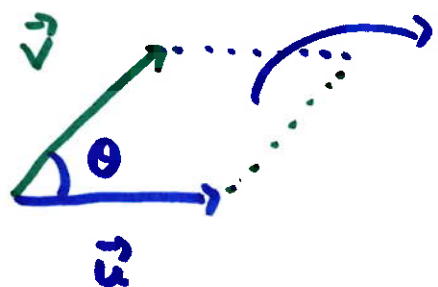
$$\vec{u} \times \vec{v} = \langle 1, 8, -5 \rangle \quad \vec{v} \times \vec{u} = -\langle 1, 8, -5 \rangle = \langle -1, -8, 5 \rangle \quad (\text{check on your own})$$

$$\text{notice } (\vec{u} \times \vec{v}) \cdot \vec{u} = \langle 1, 8, -5 \rangle \cdot \langle 2, 1, 2 \rangle = 2 + 8 - 10 = 0$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle 1, 8, -5 \rangle \cdot \langle 5, 0, 1 \rangle = 5 + 0 - 5 = 0$$

this means, $\vec{u} \times \vec{v}$ (or $\vec{v} \times \vec{u}$) is perpendicular or orthogonal to BOTH $\vec{u}$ and $\vec{v}$

if $\vec{u}$ and $\vec{v}$ form the two sides of a parallelogram sharing a corner, then we can also show that $|\vec{u} \times \vec{v}| = \text{area of the parallelogram}$

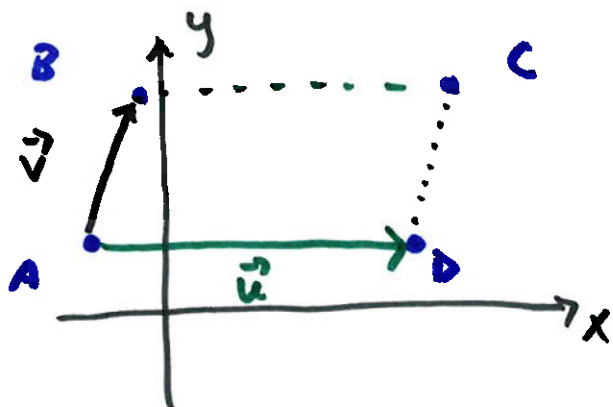


$$\text{area} = |\vec{u} \times \vec{v}|$$

$$= |\vec{u}| |\vec{v}| \sin \theta$$

example Find area of para parallelogram with vertices

$$A(-3, 4), B(-1, 7), C(3, 5), D(1, 2)$$



$$\vec{u} = \langle 4, -2 \rangle = \langle 4, -2, 0 \rangle$$

$$\vec{v} = \langle 2, 3 \rangle = \langle 2, 3, 0 \rangle$$

area of p-gram is $|\vec{u} \times \vec{v}|$ or $|\vec{v} \times \vec{u}|$

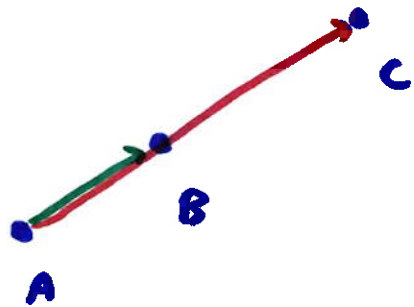
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(16) = 16\vec{k} = \langle 0, 0, 16 \rangle$$

$$|\vec{u} \times \vec{v}| = \boxed{16} \text{ area of the p-gram}$$

$$\text{or } |\vec{u}| |\vec{v}| \sin \theta$$

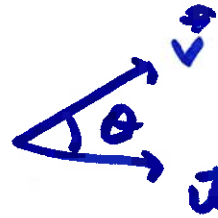
collinear vectors: along the same line



if $\vec{AB} \parallel \vec{AC}$, then angle between

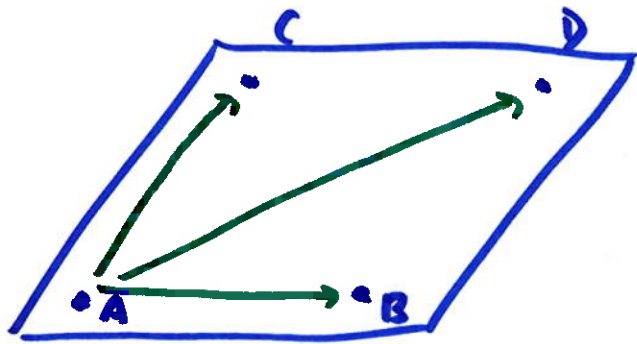
is $\theta = 0$ (same direction)

$= 180^\circ$ (opposite direction)



and $\sin \theta = 0$

therefore, if $\vec{AB} \parallel \vec{AC}$, then $|\vec{AB} \times \vec{AC}| = 0$



A, B, C, D are points on a plane

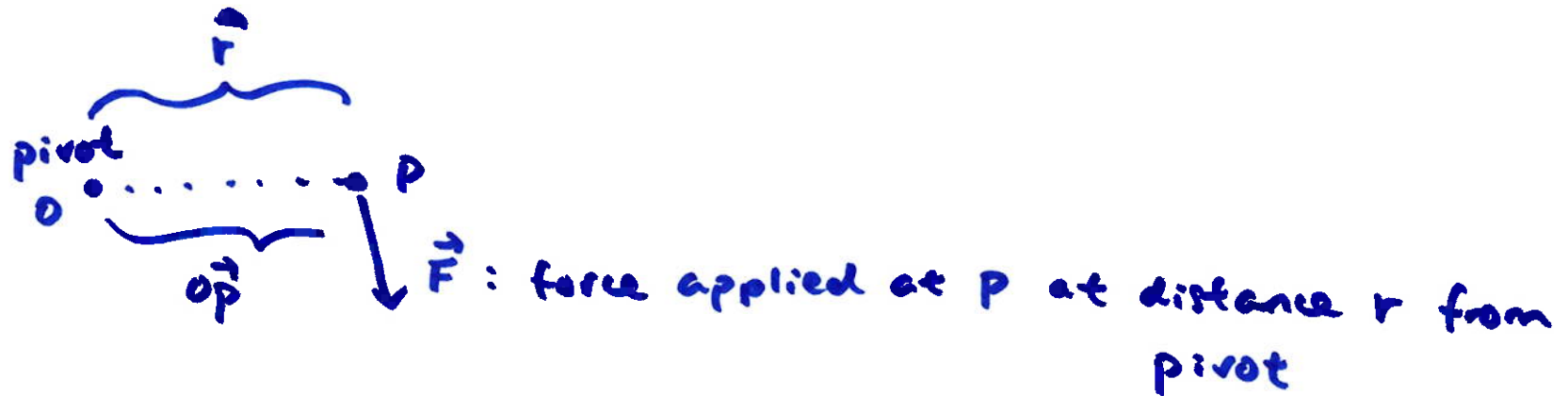
$\vec{AB} \times \vec{AD}$, $\vec{AD} \times \vec{AB}$, $\vec{AB} \times \vec{AC}$, $\vec{AC} \times \vec{AB}$

$\vec{AD} \times \vec{AC}$, $\vec{AC} \times \vec{AD}$

are ALL orthogonal to the plane

cross product is orthogonal to BOTH
parent vectors

physical applications: torque or moment can be found by cross product



the torque about pivot is $\vec{\tau} = \vec{r} \times \vec{F}$

magnetism: $\vec{F} = q (\vec{v} \times \vec{B})$

charge flow of current magnetic field