Second derivative test says

$$f_x = 0 \text{ and } f_y = 0 \Rightarrow x = a,\ y = b$$

If at (a,b) $f_{xx}f_{yy} - (f_{xy})^2 > 0$, $f_{xx} > 0$, local min

If at (a,b) $f_{xx}f_{yy} - (f_{xy})^2 > 0$, $f_{xx} < 0$, local max

If at (a,b) $f_{xx}f_{yy} - (f_{xy})^2 < 0$, saddle point.

Why?

We can use two-variable Taylor series expansion about (a,b) to rewrite $z = f(x,y)$ as

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \cdots$$

(note the first line is simply the tangent plane approximation of $f(x,y)$)

at (a,b), $f_x(a,b) = f_y(a,b) = 0$ (definition of critical points)

let's truncate the series after the second-order terms (ignore the $+\cdots$)

$$f(x,y) \approx f(a,b) + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

if $f(a,b)$ is a local minimum, then

$$f(x,y) - f(a,b) > 0$$

so $\frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] > 0$

or $f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 > 0$

let $x-a = h$, $y-b = k$

we want $f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2 > 0$

divide by k^2: $\underbrace{f_{xx}\left(\frac{h}{k}\right)^2 + 2f_{xy}\left(\frac{h}{k}\right) + f_{yy} > 0}_{\text{quadratic in the form of}}$

$$f_{xx} w^2 + 2f_{xy} w + f_{yy} > 0$$

consider $f_{xx} w^2 + 2f_{xy} w + f_{yy} = 0$ first.

its roots are: $w = \dfrac{-f_{xy} \pm \sqrt{(f_{xy})^2 - f_{xx} f_{yy}}}{f_{xx}}$

if $(f_{xy})^2 - f_{xx} f_{yy} < 0$ (or $f_{xx} f_{yy} - (f_{xy})^2 > 0$)
then there are no real roots
and $f_{xx} w^2 + 2f_{xy} w + f_{yy}$ is always positive if

$f_{xx} > 0$

so, $f(x,y) - f(a,b) = \frac{1}{2!}\left[f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right] > 0$

this is why $D = f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$ means we have a local minimum at (a,b).

and if $f_{xx} < 0$, then as long as $f_{xx}f_{yy} - (f_{xy})^2 > 0$ means there are no real roots to $f_{xx}w^2 + 2f_{xy}w + f_{yy} = 0$ and $f_{xx}w^2 + 2f_{xy}w + f_{yy} < 0$

If $f_{xx}f_{yy} - (f_{xy})^2 < 0$, then w will have real roots, and that means $f_{xx}w^2 + 2xyw + f_{yy}$ will sometimes by positive and sometimes be negative, so $f(x,y) - f(a,b)$ will sometimes be greater than zero and sometimes less than zero.