15.7 Maximum and Minimum Problems (part 1)

recall if $y = f(x)$, then $(c, f(c))$ is a <u>critical point</u> if $f'(c) = 0$

And the <u>Second Derivative Test</u> can be used to determine what that critical

point is : if $f''(c) > 0$, then there is a relative/local minimum at $x = c$

$f''(c) < 0$, " " " maximum at $x = c$

$f''(c) = 0$, then the test is inconclusive.

the two-variable version of the <u>Second Derivative Test</u> is :

if $f_x(a,b) = f_y(a,b) = 0 \rightarrow$ then (a,b) is a <u>critical point</u>

then the <u>discriminant</u> $D = f_{xx} f_{yy} - (f_{xy})^2$ at (a,b) and the sign

of f_{xx} at (a,b) determine what the critical point is

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$D = f_{xx} f_{yy} - (f_{xy})^2$

if at (a,b) $D > 0$ and $f_{xx} > 0$, then there is a local minimum at (a,b)

if at (a,b) $D > 0$ and $f_{xx} < 0$, then there is a local maximum at (a,b)

if at (a,b) $D < 0$ (and f_{xx} doesn't matter), then (a,b) is neither a local max nor a local min

→ saddle point

if at (a,b) $D = 0$, then the test is inconclusive

((a,b) could still be a local max/min or a saddle point)

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why?

Let's examine the discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$ a bit more

consider when $D > 0$, then f_{xx} and f_{yy} must have the same sign

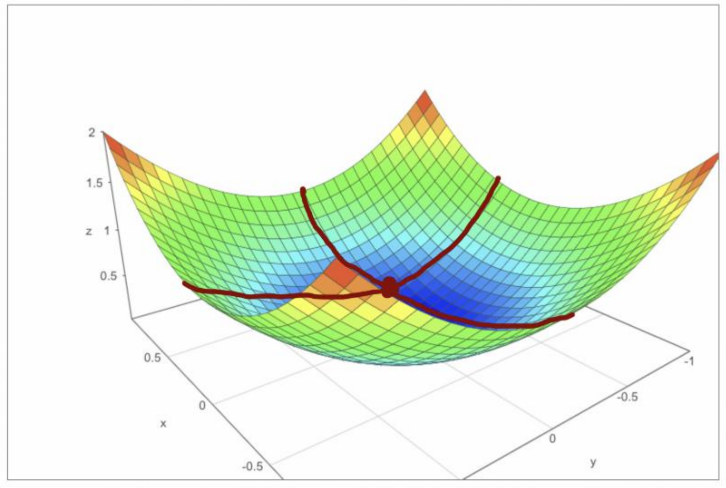
(that explains why we only need to look at the sign of f_{xx})

f_{xx} is the concavity of a slice of $f(x,y)$ with y held constant

$f_{xx} > 0 \rightarrow$ slice is concave up, $f_{xx} < 0 \rightarrow$ slice is concave down

if $D > 0$, f_{xx} and f_{yy} have the same sign, so this means the two slices that intersect at (a,b) have the same concavity, so therefore, (a,b) is either at the top of a parabolic shape or at bottom of one

$\rightarrow$ that's why $D > 0$ and $f_{xx} > 0$ ($f_{yy} > 0$) gives a local min



at the critical point, both directions look at bottom of a parabola ($f_{xx} > 0$, $f_{xx} > 0$)

same idea if $D > 0$ and $f_{xx} < 0$

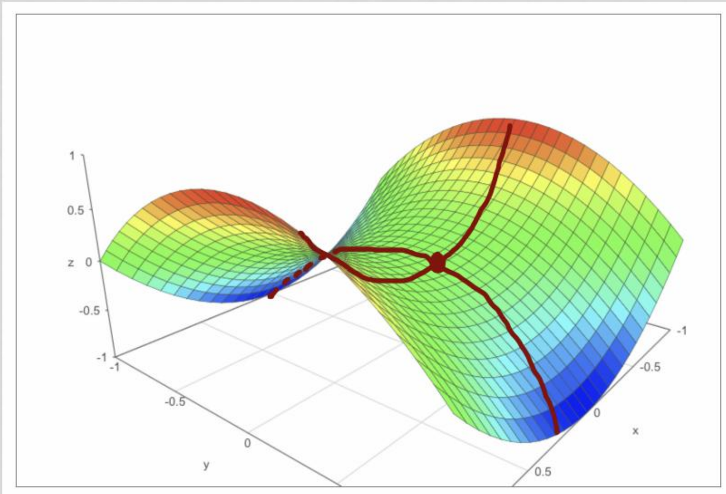
consider why $D < 0$, then $D = f_{xx} f_{yy} - (f_{xy})^2 < 0$ implies either f_{xx} and f_{yy} have opposite signs or f_{xx} or f_{yy} is zero

f_{xx} and f_{yy} opposite signs →

one direction → bottom of parabola opening up

the other direction → top of parabola opening down

→ saddle point



example $f(x,y) = x^3 - 48xy + 64y^3$ find relative max/min

find critical points: $f_x = 0$ and $f_y = 0$

$f_x = 3x^2 - 48y = 0$ — (1)

$f_y = -48x + 192y^2 = 0$ — (2)

from (1), $x^2 = 16y$ — (3)

from (2), $x = 4y^2$ — (4)

sub (4) into (3), we get $(4y^2)^2 = 16y$

$16y^4 = 16y$

$16y^4 - 16y = 0$

$16y(y^3 - 1) = 0 \rightarrow y = 0, \; y = 1$

to form critical points, we need the x values to pair up w/ them from (4) : $x = 4y^2$

if $y=0$, $x=0$
if $y=1$, $x=4$

$\boxed{\text{two critical points: } (0,0),\ (4,1)}$

each of these could be a local max/min or saddle pt
we need the discriminant to identify

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_x = 3x^2 - 48y$ $\quad f_y = -48x + 192y^2$

$f_{xx} = 6x$ $\quad f_{yy} = 384y$ $\quad f_{xy} = f_{yx} = -48$

$D = 2304xy - 2304$

at $(0,0)$, $D < 0$ $\longrightarrow$ $\boxed{\text{saddle point at } (0,0)}$

at $(4,1)$, $D > 0$ and $f_{xx} = 6x > 0$ $\longrightarrow$ $\boxed{\text{local minimum at } (4,1)}$

usually, the most difficult part is in find critical points

<u>example</u> $f(x,y) = xy e^{-x^2-y^2}$

using product rule,

$$f_x = (xy)e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(y) = 0$$

$$= ye^{-x^2-y^2}(-2x^2+1) = 0 \quad — \quad (1)$$

Similarly,

$$f_y = \cdots = xe^{-x^2-y^2}(-2y^2+1) = 0 \quad — \quad (2)$$

solve for x, y

from (1) $\underbrace{\left(e^{-x^2-y^2}\right)}_{\text{never zero}}(y)(-2x^2+1) = 0$

so, $\underline{y = 0}$ or $-2x^2+1=0$

$$\underline{x = \pm \frac{1}{\sqrt{2}}}$$

Do NOT pair these to form points!

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because they came from solving $f_x = 0$ ONLY they can make

$f_x = 0$ but CANNOT make $f_y = 0$ in general

and critical points MUST make BOTH f_x and $f_y = 0$

from 1, $y = 0$, $x = \pm \frac{1}{\sqrt{2}}$

from 2 : $x e^{-x^2-y^2}(2y^2-1) = 0$

$x = 0$, $2y^2 - 1 = 0$

$y = \pm \frac{1}{\sqrt{2}}$

do NOT pair these to form points for the same listed above we must form points from solutions of BOTH $f_x = 0$ and $f_y = 0$

$$f_x = 0 \longrightarrow \quad \boxed{\begin{matrix} y = 0, \\ x = 0 \end{matrix}} \quad \boxed{\begin{matrix} x = \pm \frac{1}{\sqrt{2}} \\ y = \pm \frac{1}{\sqrt{2}} \end{matrix}}$$

$$f_y = 0 \longrightarrow$$

one critical point

$(0, 0)$

four critical points

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

we have FIVE critical points

now we find $D = f_{xx} f_{yy} - (f_{xy})^2$

$$f_{xx} = \cdots = 2xy(2x^2 - 3)e^{-x^2 - y^2}$$

$$f_{yy} = \cdots = 2xy(2y^2 - 3)e^{-x^2 - y^2}$$

$$f_{xy} = \cdots = (2x^2 - 1)(2y^2 - 1)e^{-x^2 - y^2}$$

now test at all five critical pts

at $(0,0)$ $f_{xx}=0$, $f_{yy}=0$, $f_{xy}=1$, so $D<0$ saddle point

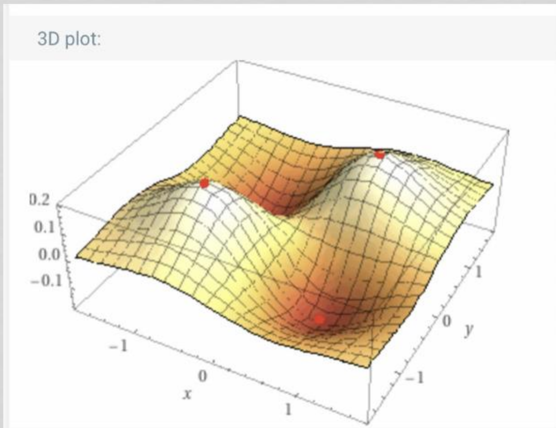
at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $f_{xx}=-2e^{-1}$, $f_{yy}=-2e^{-1}$, $f_{xy}=0$, so $D>0$, $f_{xx}<0$

relative maximum

Similarly, at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, there is a relative maximum

at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, there is a relative minimum

at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, there is a relative minimum



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what do we do when $D = 0$? Test fails but we still need to determine the nature of the critical point.

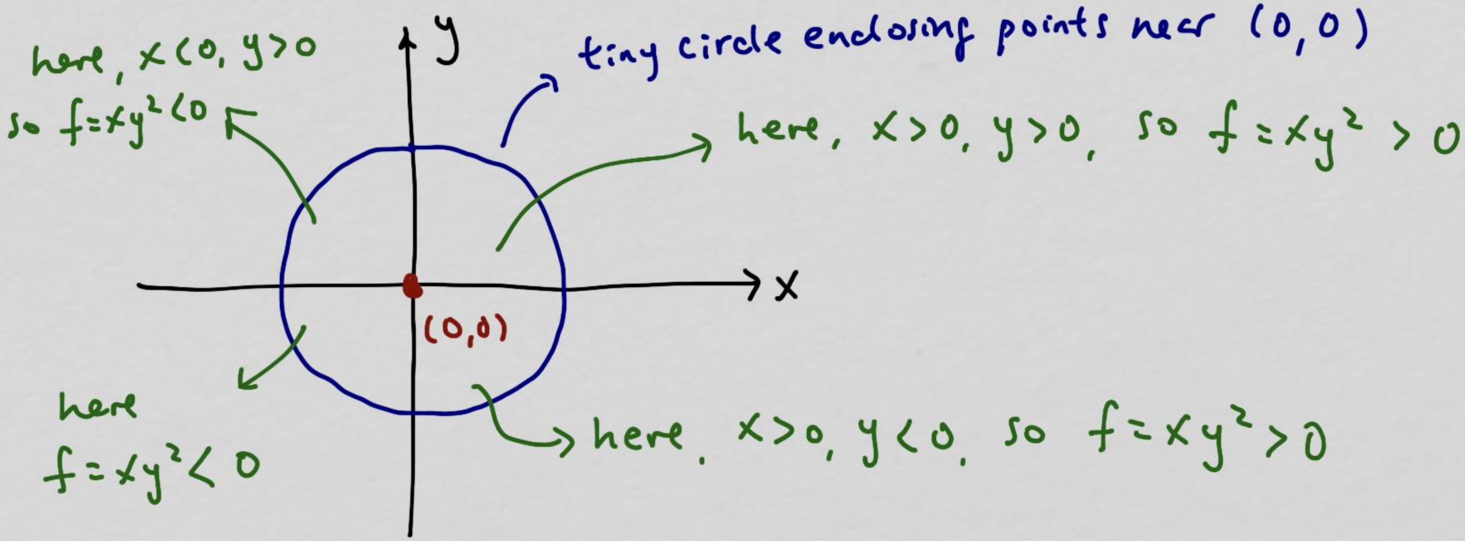
<u>example</u> $f(x,y) = xy^2$

$(0,0)$ is a critical point

$f_x = y^2 \quad f_y = 2xy \quad f_{xx} = 0, \quad f_{yy} = 2x \quad f_{xy} = 2y$

$D = -4y^2 = 0$ at $(0,0)$ $\rightarrow$ Test is inconclusive

let's examine the values of $f(x,y)$ at points near $(0,0)$



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if $f(0,0)=0$ is a local maximum, then we expect $f=xy^2$ to be all less than 0 in all quadrants near $(0,0)$

if $f(0,0)=0$ is a local minimum, then we expect $f=xy^2$ to be all greater than 0 in all quadrants near $(0,0)$

but we have neither of the above

here, going toward positive x f is increasing, but going toward negative x f is decreasing, so $(0,0)$ cannot be the location of a local max or a local min

$\rightarrow$ it's a saddle point