# 15.8 Lagrange Multipliers

now we look at constrained optimization problems – max/min something while satisfying some condition

for example, max/min of $f(x,y) = x^2+y^2$ subject to the condition $xy = 1$

this means find $(x,y)$ such that $f(x,y) = x^2+y^2$ is as large/small as possible while satifying $xy = 1$

here, $f(x,y) = x^2+y^2$ is called the objective (the goal)

$\underbrace{g(x,y) = xy-1 = 0}_{xy=1 \text{ written differently}}$ is called the constraint (the condition) one way to solve is by simple substitution

$$f(x,y) = x^2 + y^2 \qquad xy = 1 \rightarrow y = \frac{1}{x}$$

$$f(x) = x^2 + \left(\frac{1}{x}\right)^2$$

$$f = x^2 + \frac{1}{x^2}$$

$$f' = 2x - \frac{2}{x^3} = 0 \longrightarrow 2x = \frac{2}{x^3}$$

$$x^4 = 1 \rightarrow x = 1, -1$$

the corresponding y's are: $y = \frac{1}{1} = 1$, $y = \frac{1}{-1} = -1$

critical points: $(1,1)$, $(-1,-1)$

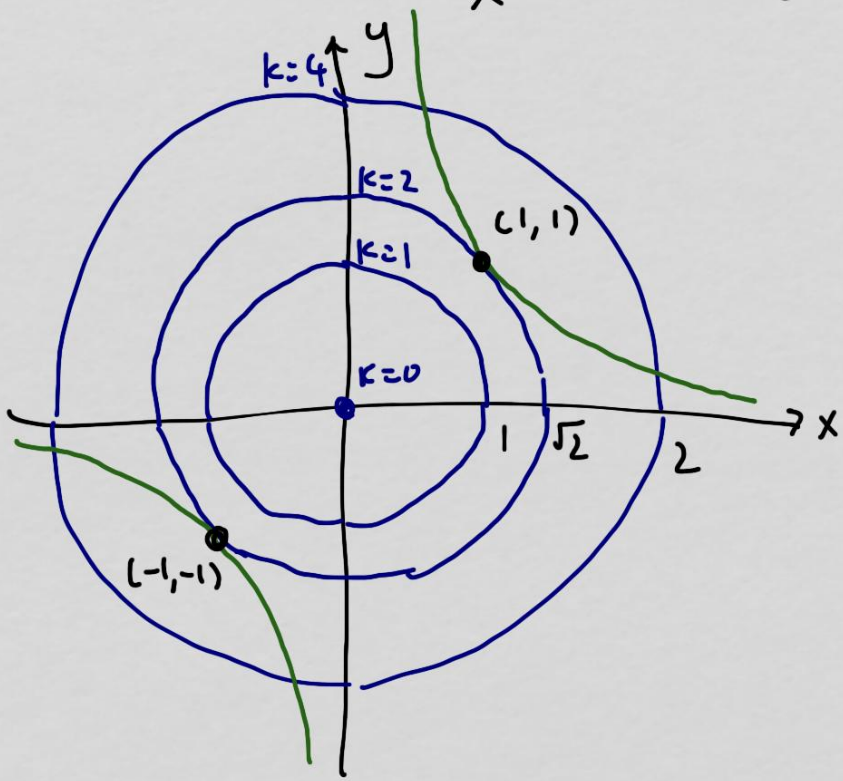
$f(x,y) = x^2 + y^2$ with $xy = 1$

does not have a maximum, because we can make $x$ or $y$ arbitrarily large (and therefore making $f$ arbitrarily large) and the other variable will always be defined through $xy = 1$

so, by that logic, the critical pts $(1, 1)$ and $(-1,-1)$ must be locations of minimum $f$ $\left( f(1,1) = f(-1,-1) = 2 \right)$

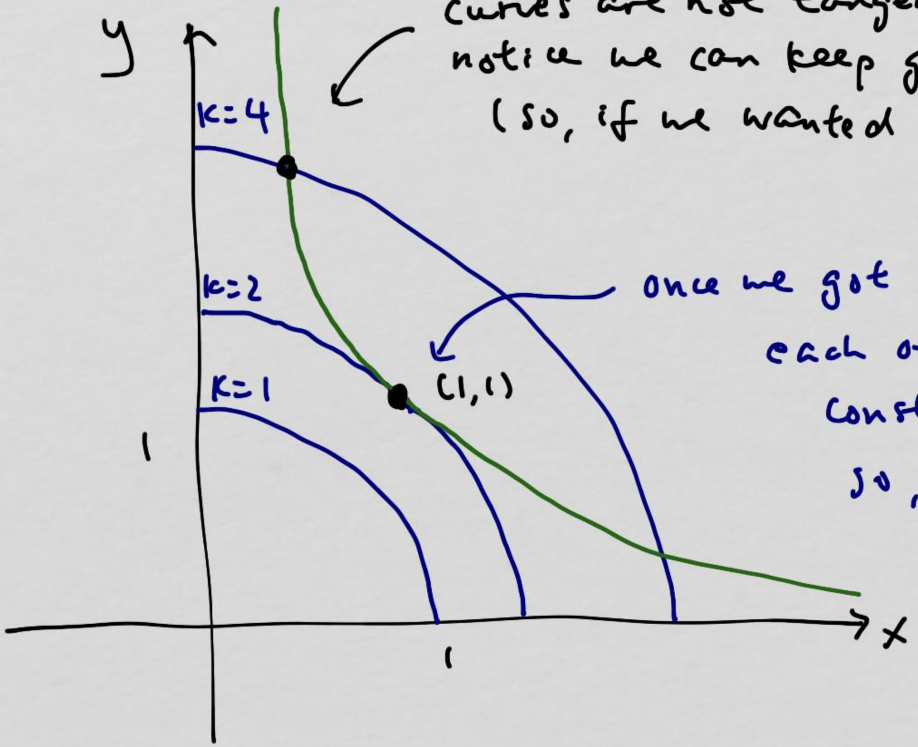
now the important part: the geometric interpretation

sketch a few level curves of $f(x,y) = x^2+y^2 = k$ and the constraint $y = \frac{1}{x}$ (from $xy = 1$)




we want to find max/min $k$ while only moving along on the constraint curve (green curve)

notice that the locations of minimum $k$, the level curve and the constraint curve are tangent to each other being tangent to each other means any more movement on the constraint in either direction will result in higher $k$

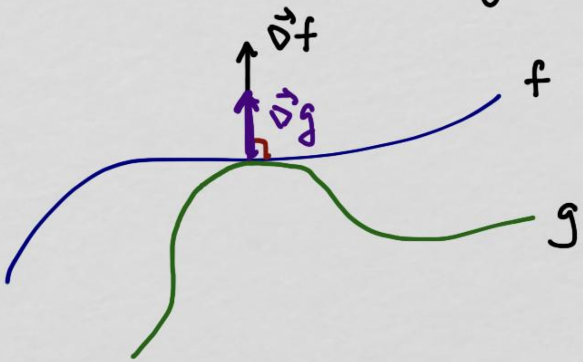



curves are not tangent to each other
notice we can keep going down to find lower $k$
(so, if we wanted minimum $k$, don't stop here)

once we got here (where curves are tangent to each other), any movement along the constraint results in higher $k$
so, if we wanted minimum $k$, we stop here.

if we can find where on $g(x,y)=0$ the place where it's tangent to a level curve, then we have potential locations of max/min $f(x,y)$

if two curves are tangent to each other, then their gradient vectors are also tangent to each other




this means at the locations where the level curve of $f(x,y)$ is tangent to the constraint curve, the gradient of $f$ must be some constant multiple of the gradient of $g$

Greek letter lambda

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

Lagrange Multiplier

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to find locations of max/min of $f(x,y)$ subject to the constraint $g(x,y)=0$, we solve $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x,y)=0$ simultaneously for $x, y$, and sometimes $\lambda$. Then evaluate and compare $f(x,y)$ at these locations to find max/min.

this procedure is called the <u>Method of Lagrange Multipliers</u>

<u>example</u> Find max/min of $f(x,y) = 4 - x^2 - y^2$ subject to the constraint $4x^2 + y^2 = 4$

first, rewrite the constraint : $g(x,y) = 4x^2 + y^2 - 4 = 0$

then $\vec{\nabla} f = \langle -2x, -2y \rangle \quad \vec{\nabla} g = \langle 8x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$\langle -2x, -2y \rangle = \lambda \langle 8x, 2y \rangle$

equating the components, we get

$$-2x = \lambda \cdot 8x \quad - \quad (1)$$

$$-2y = \lambda \cdot 2y \quad - \quad (2)$$

from (1) $\quad \lambda \cdot 8x + 2x = 0$

$2x(4\lambda + 1) = 0 \rightarrow x = 0, \quad \lambda = -1/4$

sub $x=0$ into $g(x,y) = 4x^2 + y^2 - 4 = 0$

$y^2 - 4 = 0 \rightarrow y = 2, -2$

so we get two critical points : $\boxed{(0, 2), \ (0, -2)}$

from (2) $\quad \lambda \cdot 2y + 2y = 0$

$2y(\lambda + 1) = 0 \rightarrow y = 0, \quad \lambda = -1$

sub $y=0$ into $g(x,y) = 4x^2 + y^2 - 4 = 0$

$4x^2 - 4 = 0 \rightarrow x = 1, -1$

two more critical points: $\boxed{(1, 0), (-1, 0)}$

if we take $\lambda = -1/4$ ( from solving ① )

into ②, we get $-2y = (-1/4) \cdot 2y$

$$-2y = -\tfrac{1}{2}y \longrightarrow y = 0$$

put into $g(x,y) = 4x^2 + y^2 - 4 = 0$

we get $x = \pm 1$

so we get the critical points

$(1, 0)$, $(-1, 0)$ which we already have

so, the $\lambda$ values don't add any new information here

now we compare $f(x,y) = 4 - x^2 - y^2$ at the critical points

$$\left.\begin{array}{l} f(0,2) = 0 \\ f(0,-2) = 0 \end{array}\right\} \text{min}$$

$$\left.\begin{array}{l} f(1,0) = 3 \\ f(-1,0) = 3 \end{array}\right\} \text{max}$$

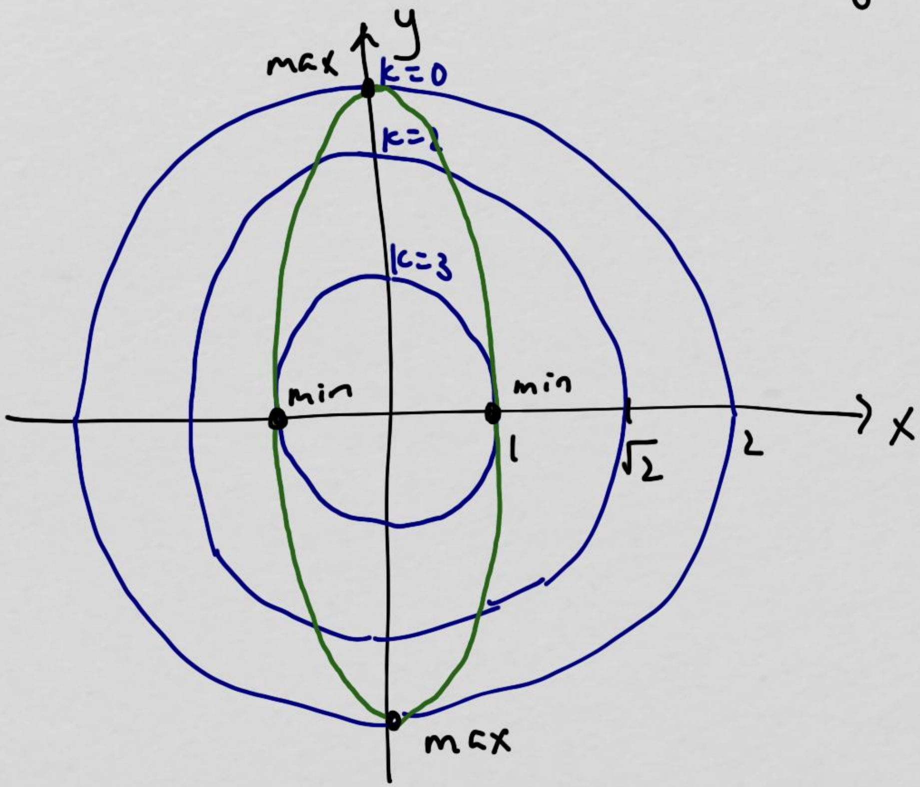
graph: $f(x,y) = 4 - x^2 - y^2$ level curves at $f(x,y)=k$ are

$$x^2+y^2 = 4-k$$

circles of radius $\sqrt{4-k}$

$g(x,y) = 4x^2+y^2 = 4$ ellipse x-ints : $\pm 1$

y-ints: $\pm 2$



**example** $f(x,y,z) = xyz$

$$g(x,y,z) = x^2+y^2+z^2-3 = 0$$

geometrically, we want to find points on $x^2+y^2+z^2=3$ (sphere of radius $\sqrt{3}$) where the product of the $x, y, z$ values are max/min.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\left.\begin{aligned} yz &= \lambda \cdot 2x \quad - (1) \longrightarrow \lambda = \frac{yz}{2x} \\ xz &= \lambda \cdot 2y \quad - (2) \longrightarrow \lambda = \frac{xz}{2y} \\ xy &= \lambda \cdot 2z \quad - (3) \longrightarrow \lambda = \frac{xy}{2z} \end{aligned}\right\} \text{imply } \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

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$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$y^2 z = x^2 z$ $\qquad$ $z^2 x = x y^2$ or $y^2 = z^2$ — (5)

or $x^2 = y^2$ — (4)

(4) and (5) together say $\boxed{x^2 = y^2 = z^2}$

Sub into $g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$

$x^2 + x^2 + x^2 = 3$

$3x^2 = 3$

$x = \pm 1$

since $y^2 = x^2$, $\quad y = \pm 1$

since $z^2 = y^2$ $\quad z = \pm 1$

give us 8 critical points

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the 8 critical points are: $(1,1,1)$

$(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$

$(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)$

$(-1,-1,-1)$

Compare $f(x,y,z) = xyz$ at these points

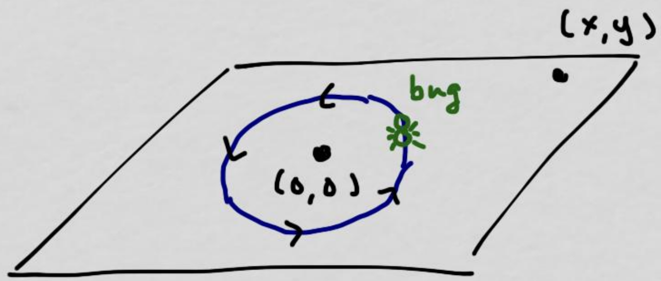
$f = 1$ at the first 4 points

$f = -1$ at the last 4 points

$f$ has a max of $1$ and a min of $-1$

<u>example</u> The temperature on a metal plate is

$T(x,y) = x^2 - 4xy + 4y^2$ where $(0,0)$ is the center of the plate. If a bug is walking around in a circle of radius $\sqrt{5}$ around the center of the plate, find the maximum temperature the bug experiences.




has temperature
$T(x,y) = x^2 - 4xy + 4y^2$

this is equivalent to find max/min of $T(x,y) = x^2 - 4xy + 4y^2$

subject to $g(x,y) = \underbrace{x^2 + y^2 - 5 = 0}$

circle $x^2 + y^2 = 5$
$(x,y)$ must be part of the circle

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$$\vec{\nabla} T = \lambda \vec{\nabla} g$$

$$\langle 2x-4y,\ 8y-4x \rangle = \lambda \langle 2x, 2y \rangle$$

$2x - 4y = \lambda \cdot 2x$ — (1)

multiply (1) by -2

$-4x + 8y = \lambda \cdot 2y$ — (2)

$-4x + 8y = \lambda \cdot -4x$ — (3)

give us $\lambda \cdot 2y = \lambda \cdot -4x$ since their left sides are the same

$$\lambda \cdot 2y = \lambda \cdot -4x$$

$$4\lambda x + 2\lambda y = 0$$

$2\lambda(2x+y) = 0 \longrightarrow \boxed{y=-2x}$ or $\boxed{\lambda = 0}$

if $\lambda = 0$, (1) tells us $2x = 4y$ or $\boxed{x = 2y}$

Sub into $g(x,y) = x^2+y^2-5=0$

we get $4y^2+y^2=5 \rightarrow y = \pm 1$

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$y = \pm 1$ along with $\underbrace{x = 2y}$ gives us the critical points $\boxed{(2,1), \quad (-2,-1)}$

$x, y$ have the same sign

back to $y = -2x$, sub into $g(x,y) = x^2 + y^2 - 5 = 0$

we get $x^2 + (-2x)^2 = 5$

$5x^2 = 5 \qquad x = \pm 1$ along with $\underbrace{y = -2x}$

implies $x, y$ have opposite signs

we get two more critical points: $\boxed{(1,-2), \ (-1, 2)}$

now compare $T(x,y) = x^2 - 4xy + 4y^2$ at the critical pts

$$
\begin{aligned}
T(2,1) &= 0 \\
T(-2,-1) &= 0 \\
T(1,-2) &= 13 \\
T(-1,2) &= 13
\end{aligned}
$$

max temperature the bug experiences