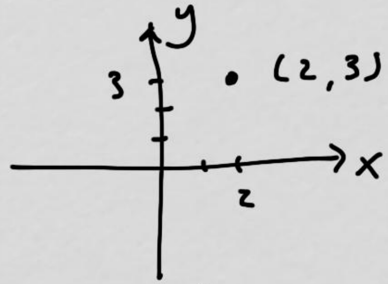
16.3 Double Integrals in Polar Coordinates

Rectangular / Cartesian coordinates : $(2, 3)$

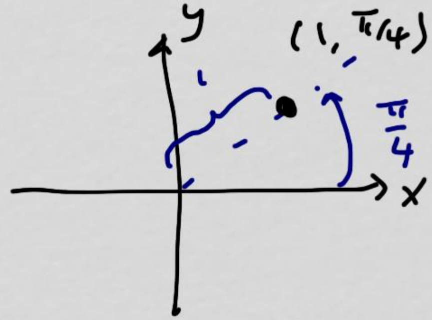
x $\quad$ y



Polar coordinates : $\left(1, \frac{\pi}{4}\right)$

r — displacement from origin

θ — angle with respect to positive x-axis



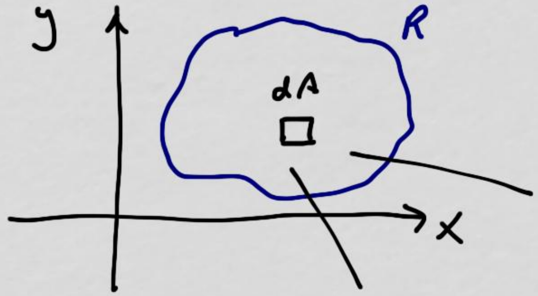
Conversion:

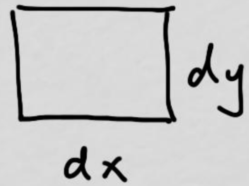
$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

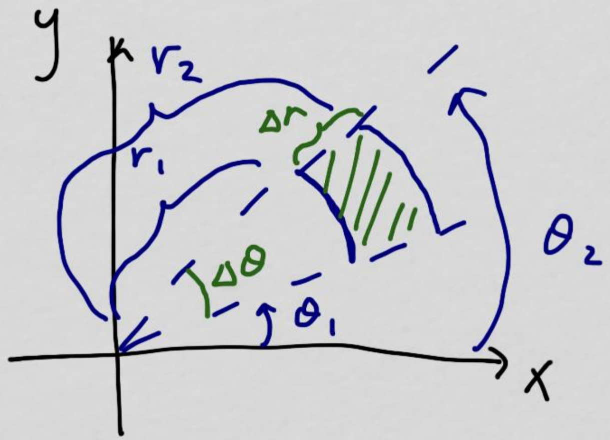
In Cartesian, $\iint_R f(x,y)\,dA$





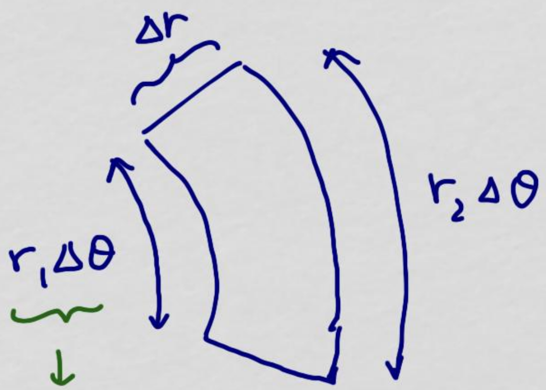
$$dA = dx\,dy$$
$$= dy\,dx$$

In Polar, $\iint_R f(r,\theta)\,\underline{\underline{dA}} \; ?$



the pizza crust region is dA:

area = ?



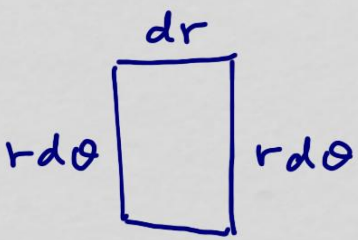
circular arc with radius r_1
angle $\Delta\theta$

not a rectangle, so area is not as easy to find

but as we shrink $\Delta r \to dr$ and $\Delta\theta \to d\theta$
(very small change in r and θ)

then $r_1 \approx r_1 = r$

so, the pizza crust becomes roughly a rectangle



$$dA = (r\,d\theta)(dr)$$

or

$$\boxed{dA = r\,dr\,d\theta}$$

for Polar coordinates

Integrating in Polar is especially effective when R is a part of a circle or circle-like shapes.

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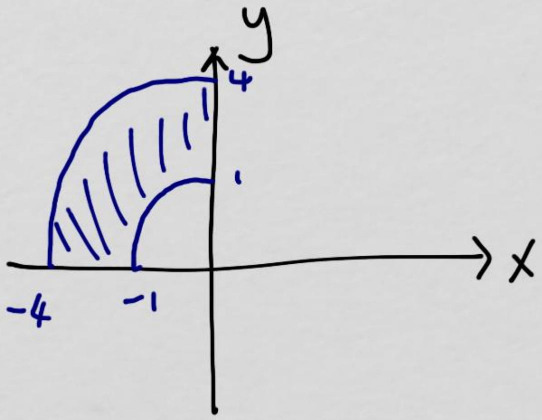
example $\iint_R \cos(x^2+y^2)\,dA$

R: region in the 2nd quadrant between two circles centered at the origin and radii 1 and 4.



$\cos(x^2+y^2)$ is hard to integrate in Cartesian

and R is a circle-like region

both of these are good reasons to switch to Polar.

describe R in Polar: $R = \{(r,\theta) : \frac{\pi}{2} \le \theta \le \pi, \quad 1 \le r \le 4\}$

express the integrand in Polar: $x^2+y^2 = r^2$

$$\cos(x^2+y^2) = \cos(r^2)$$

so, $\iint_R \cos(x^2+y^2)\,dA$

$$= \int_{\cancel{0}\ \frac{\pi}{2}}^{\cancel{\pi/2}\ \pi} \int_1^4 \cos(r^2)\, \overbrace{r\,dr\,d\theta}^{dA} = \int_{\cancel{0}\ \frac{\pi}{2}}^{\cancel{\pi/2}\ \pi} \int_1^4 r\cos(r^2)\,dr\,d\theta$$

θ $\quad$ r

I made a mistake in the video. Sorry!

$u = r^2$

$du = 2r\,dr$

$$= \int_{\cancel{0}\ \pi/2}^{\cancel{\pi/2}\ \pi} \int_1^{16} \frac{1}{2}\cos(u)\,du\,d\theta = \int_{\cancel{0}\ \pi/2}^{\cancel{\pi/2}\ \pi} \frac{1}{2}\sin(u)\Big|_1^{16} d\theta$$

$$= \int_{\cancel{0}\ \pi/2}^{\cancel{\pi/2}\ \pi} \left[\frac{1}{2}\sin(16) - \frac{1}{2}\sin(1)\right] d\theta = \frac{\sin(16)-\sin(1)}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{4}\left(\sin(16)-\sin(1)\right)}$$

still correct

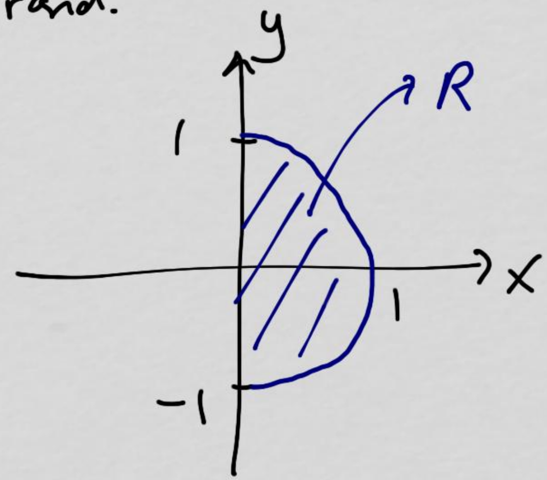
example $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)^{3/2}\,dx\,dy$

looks terrible in Cartesian, and the parts x^2+y^2 and $\sqrt{1-y^2}$ strongly suggest R is circular or circle-like and that Polar is good for the integrand.

$$-1 \le y \le 1$$

$$0 \le x \le \underbrace{\sqrt{1-y^2}}$$

$$x=\sqrt{1-y^2}$$
$$x^2+y^2=1$$



in Polar, $R=\left\{(r,\theta): -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le r\le 1\right\}$

$$(x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3$$

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integral becomes $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 (r^2)^{3/2} \underbrace{\overbrace{r\,dr\,d\theta}^{dA}}_{=}$$

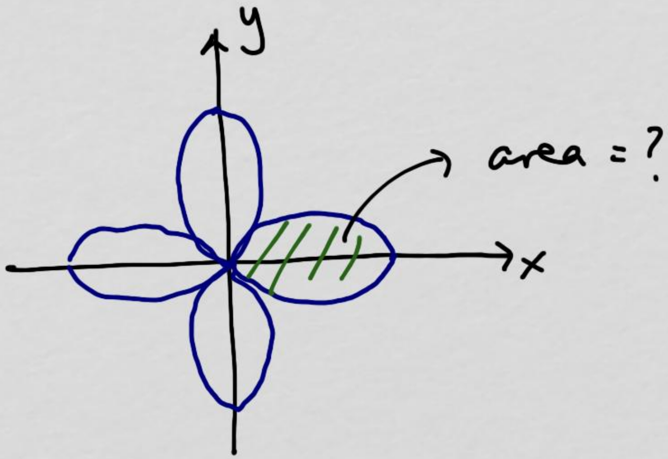
$$\theta \qquad r$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^4\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^5}{5}\Big|_0^1 d\theta$$

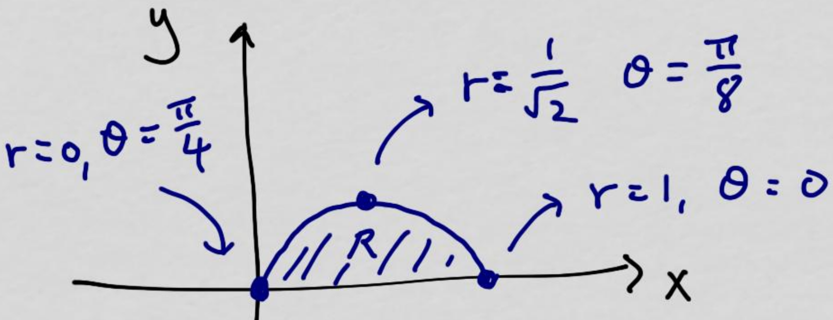
$$= \frac{1}{5}\theta\Big|_{-\pi/2}^{\pi/2} = \frac{1}{5}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{\frac{\pi}{5}}$$

other regions described by Polar equations, not just circles, are also effectively handled this way.

example Area of one petal of the rose $r = \cos 2\theta$



due to symmetry, find $\frac{1}{2}$ of the petal then multiply by 2



$0 \le \theta \le \frac{\pi}{4}$

$0 \le r \le \cos 2\theta$

$$\text{area} = \iint_R dA$$

$$= \int_0^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} r^2 \Big|_0^{\cos 2\theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2} \underbrace{\cos^2 2\theta}\, d\theta$$

$$\cos^2(u) = \frac{1+\cos 2(u)}{2}$$

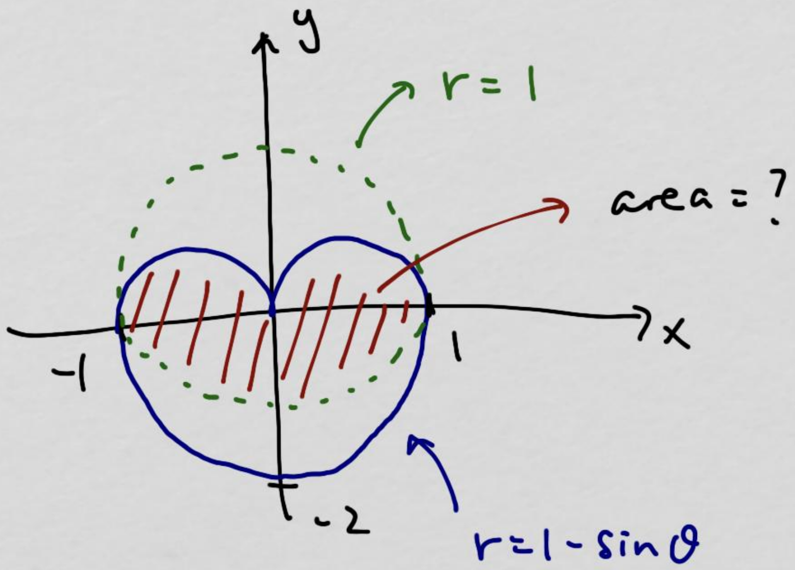
$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \cdot \frac{1+\cos(4\theta)}{2} d\theta = \frac{1}{4} \int_0^{\frac{\pi}{4}} 1+\cos(4\theta)\, d\theta$$

$$= \frac{1}{4} \left(\theta + \frac{1}{4} \sin(4\theta) \right) \Big|_0^{\pi/4} = \frac{1}{4} \left(\frac{\pi}{4} \right) = \frac{\pi}{16}$$

this is just the upper half of one petal

the area of the whole petal is twice of that: $\boxed{\frac{\pi}{8}}$

<u>example</u> Find the area of the region inside both $\underbrace{r = 1 - \sin\theta}_{\text{Cardioid}}$ and $\underbrace{r = 1}_{\text{circle radius } 1}$



θ	r
0	$\vdots$
$\frac{\pi}{4}$	$\vdots$
$\frac{\pi}{2}$	1
$\vdots$	

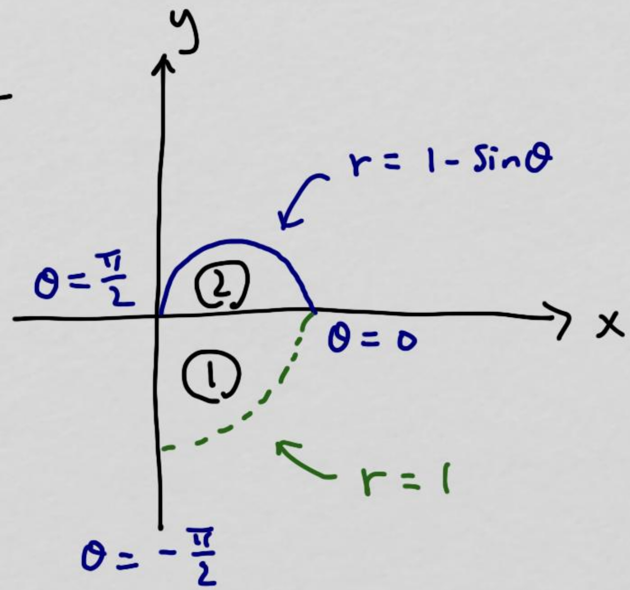
Cardioid

circle radius 1

let's find the right half then multiply by 2

note that how far from the origin we can
go in each of the two parts is
described by different r's

$\longrightarrow$ two integrals



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1 : $-\frac{\pi}{2} \le \theta \le 0$

$0 \le r \le 1$ ← circle

$$\int_{-\frac{\pi}{2}}^{0} \int_{0}^{1} r\,dr\,d\theta$$

2 : $0 \le \theta \le \frac{\pi}{2}$

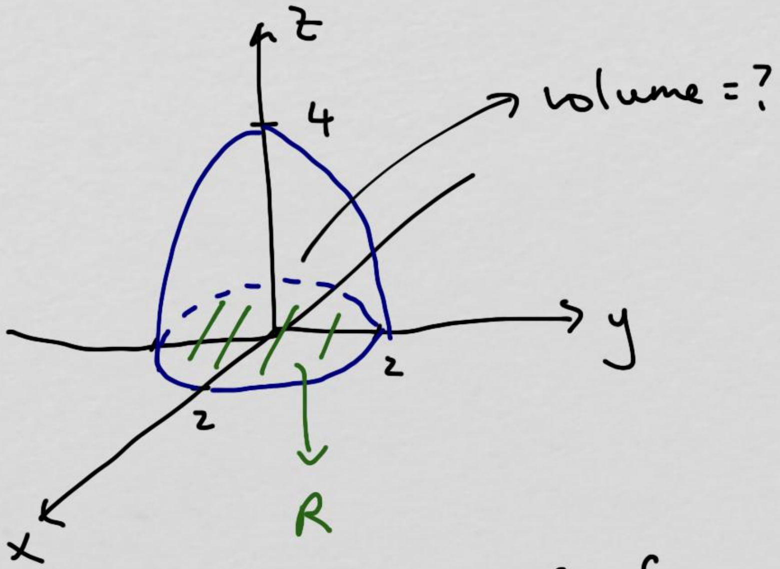
$0 \le r \le \underbrace{1-\sin\theta}_{\text{cardioid}}$

$$\int_{0}^{\frac{\pi}{2}} \int_{0}^{1-\sin\theta} r\,dr\,d\theta$$

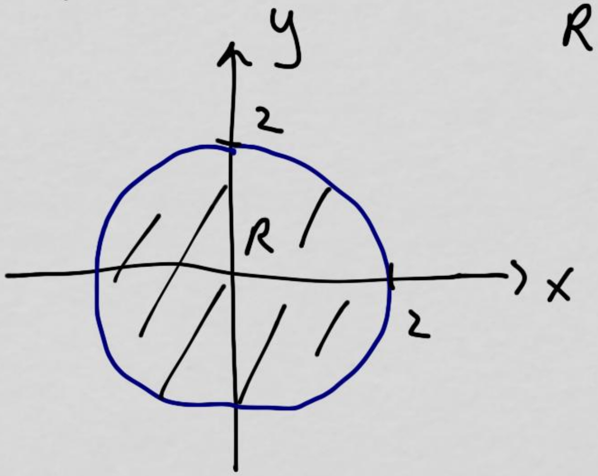
total area : $\int_{-\frac{\pi}{2}}^{0} \int_{0}^{1} r\,dr\,d\theta + \int_{0}^{\frac{\pi}{2}} \int_{0}^{1-\sin\theta} r\,dr\,d\theta$

then multiply by 2 because this is just the right half

<u>example</u> Find volume bounded by $z = 4 - x^2 - y^2$ and the xy-plane



integrate the height of the space over the "shadow" of the object (projection onto the xy-plane)



$R = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$

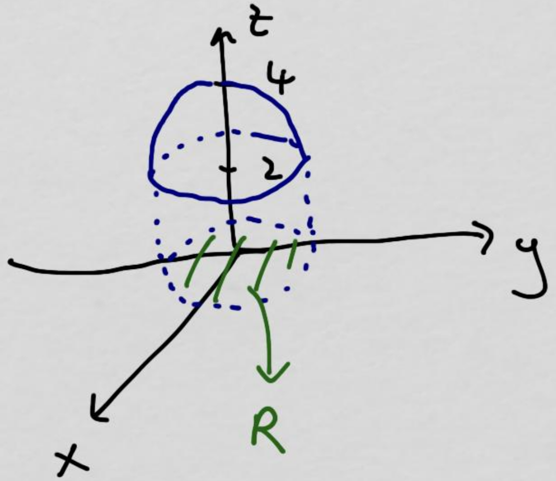
the height is $z = 4 - (x^2+y^2) = 4 - r^2$

$$V = \int_0^{2\pi} \int_0^{2} (4-r^2)\, \overbrace{r\,dr\,d\theta}^{dA}$$

$=$

$$V = \int_0^{2\pi} \int_0^2 (4r - r^3)\,dr = \int_0^{2\pi} 2r^2 - \frac{1}{4} r^3 \Big|_0^2 = \int_0^{2\pi} 4 \, d\theta = 4\theta \Big|_0^{2\pi} = \boxed{8\pi}$$

what if we only want the portion of the space above $z = 2$?



we still integrate the height over the projection onto the xy-plane

what is R ? $z = 4 - r^2$

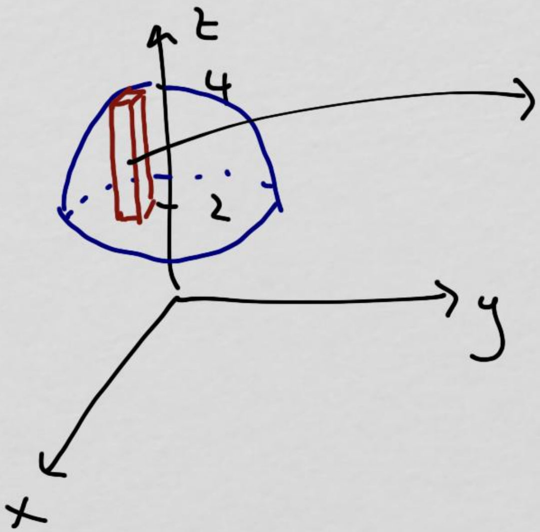
at $z = 2$, $r^2 = 2$

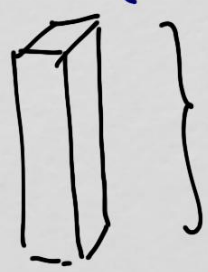
so the intersection w/ $z = 2$

(and R) is circle radius $\sqrt{2}$

$$R = \{(r, \theta) : \ 0 \le \theta \le 2\pi, \ \ 0 \le r \le \sqrt{2}\}$$

the height is also different





top is $z = 4 - r^2$

so, height is $4 - r^2 - 2 = \underline{\underline{2 - r^2}}$

bottom is $z = 2$

integrate this over R

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (2 - r^2)\, \underline{\underline{r}}\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} (2r - r^3)\, dr\, d\theta$$

$$= \cdots = \boxed{\frac{8\sqrt{2}\,\pi}{3}}$$