16.5 Triple Integrals in Spherical Coordinates

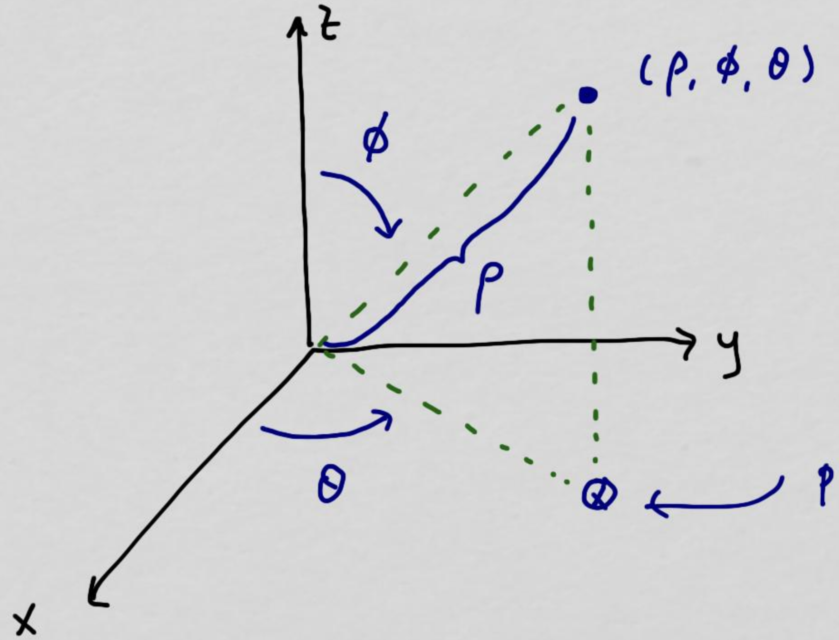
in spherical coordinates we locate a point by

$$(\rho, \phi, \theta)$$

same θ as in polar / cylindrical

"rho"
distance from the origin to the point
$\rho \geq 0$

"phi" angle measured from the positive z-axis to the point

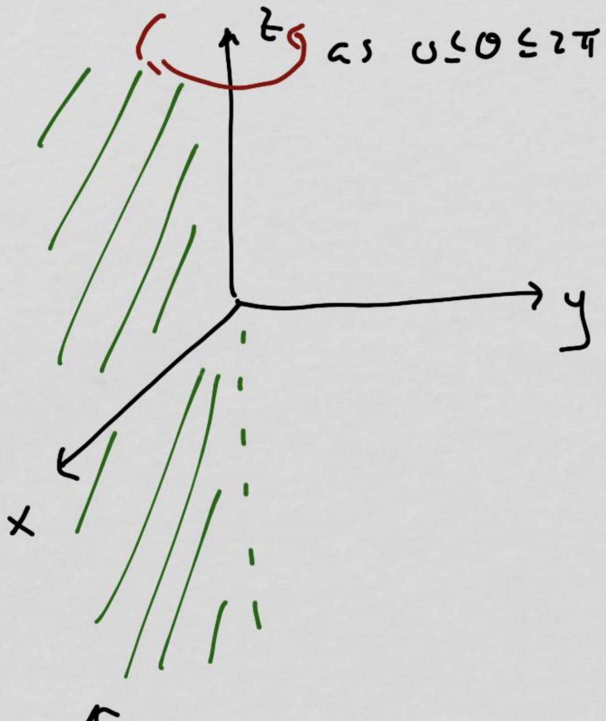


projection onto xy-plane

$$\boxed{\begin{gathered}\rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi\end{gathered}}$$

why doesn't ϕ go to 2π?

if we keep θ fixed, then $\rho \geq 0$ and $0 \leq \phi \leq \pi$ can cover half of a plane containing z-axis

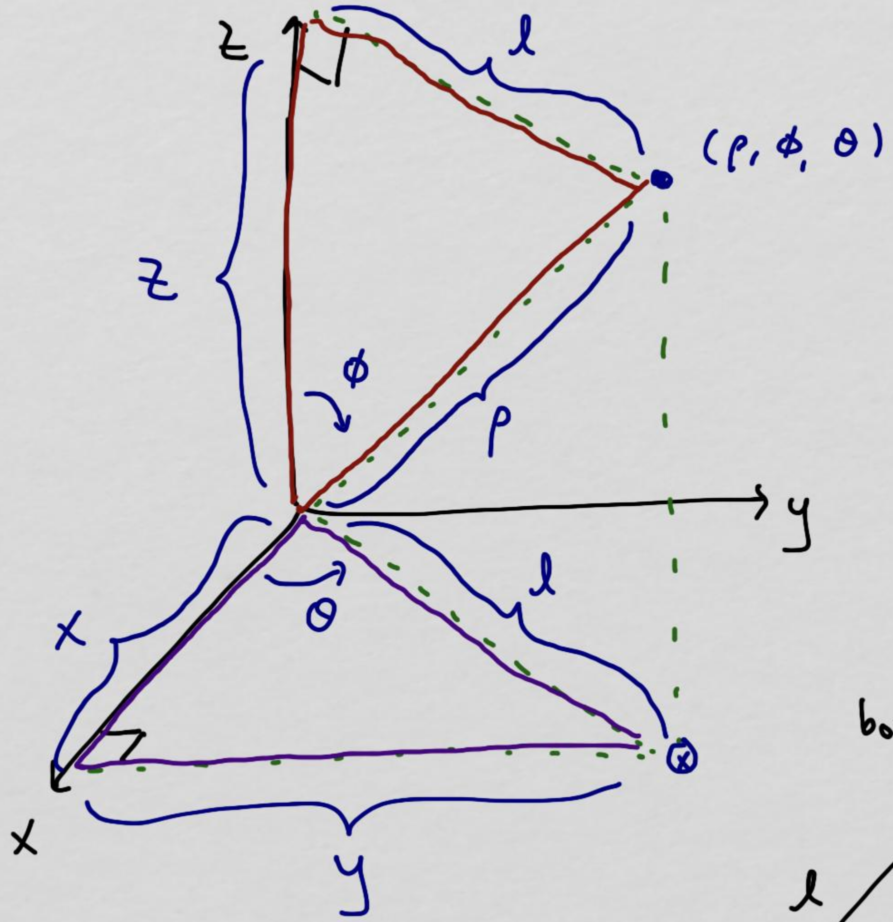


covered by $\rho \geq 0$ and $0 \leq \phi \leq \pi$

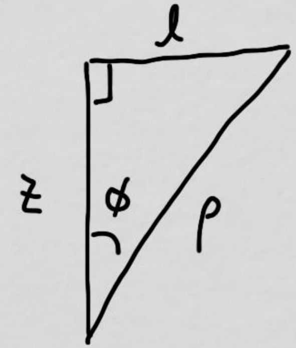
since θ can go to 2π, this let's us sweep the plane about z-axis to cover the entire $\mathbb{R}^3$

if we had let ϕ go to 2π, then we'd be covering the space twice as $0 \leq \theta \leq 2\pi$

conversion : $(\rho, \phi, \theta) \longleftrightarrow (x, y, z)$



from the top (red) triangle

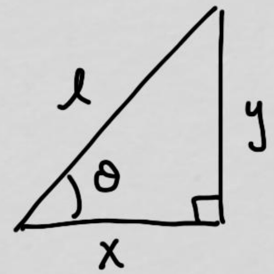


$$\cos\phi = \frac{z}{\rho} \rightarrow \boxed{z = \rho\cos\phi}$$

we also see $\sin\phi = \frac{l}{\rho}$

$$l = \rho\sin\phi$$

bottom (purple) triangle



$$\cos\theta = \frac{x}{l} = \frac{x}{\rho\sin\phi}$$

$$\boxed{x = \rho\sin\phi\cos\theta}$$

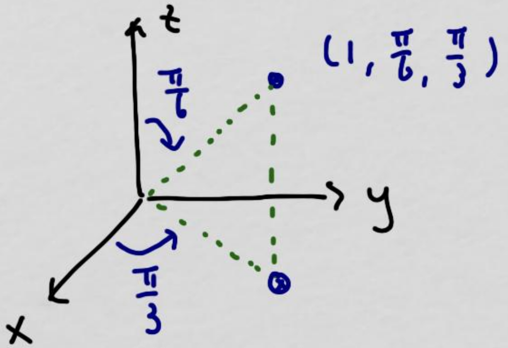
$$\sin\theta = \frac{y}{l} = \frac{y}{\rho\sin\phi}$$

$$\boxed{y = \rho\sin\phi\sin\theta}$$

$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$

note : $\boxed{x^2 + y^2 + z^2 = \rho^2}$

<u>example</u> $(\rho, \phi, \theta) = \left(1, \frac{\pi}{6}, \frac{\pi}{3}\right) \qquad (x, y, z) = ?$



by inspection, x, y, z all positive

$x = \rho \sin\phi \cos\theta = (1) \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{3}\right) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$y = \rho \sin\phi \sin\theta = (1) \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{3}\right) = 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

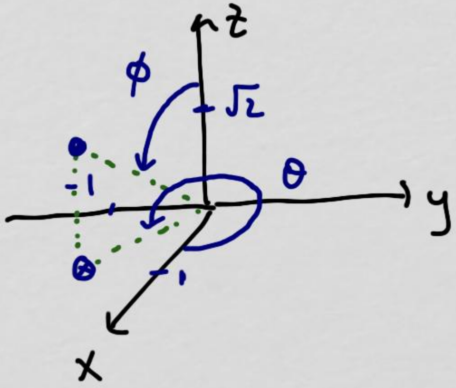
$z = \rho \cos\phi = (1) \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

check: is $x^2 + y^2 + z^2 = \rho^2$?

$$\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{16} + \frac{3}{16} + \frac{3}{4} = 1 = (1)^2 \quad \checkmark$$

therefore, $\boxed{(x, y, z) = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)}$

<u>example</u> $(x, y, z) = (1, -1, \sqrt{2})$ $(\rho, \phi, \theta) = ?$



by inspection, θ is in 4th quadrant
ϕ is in 1st quadrant

ρ is the easiest: $x^2 + y^2 + z^2 = \rho^2$

$$\rho^2 = 1^2 + (-1)^2 + (\sqrt{2})^2 = 4 \rightarrow \boxed{\rho = 2}$$

then $z = \rho \cos\phi$

$$\sqrt{2} = 2\cos\phi \rightarrow \cos\phi = \frac{\sqrt{2}}{2} \rightarrow \boxed{\phi = \frac{\pi}{4}}$$

θ is the most complicated one

$$\left.\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \end{aligned}\right\} \quad \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

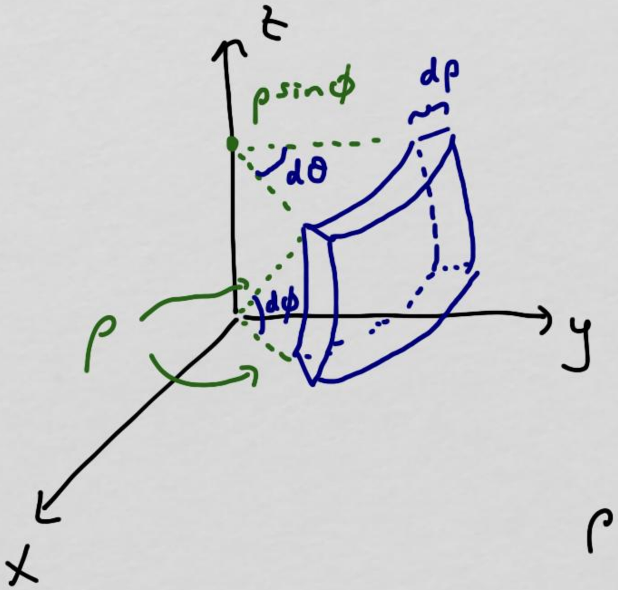
$$\tan\theta = \frac{-1}{1} = -1 \longrightarrow \theta = \tan^{-1}(-1) = \cancel{\frac{3\pi}{4}} \text{ or } \frac{7\pi}{4} \qquad \boxed{\theta = \frac{7\pi}{4}}$$

quadrant 2

need to resolve the quadrant ambiguity for θ by visual inspection of location

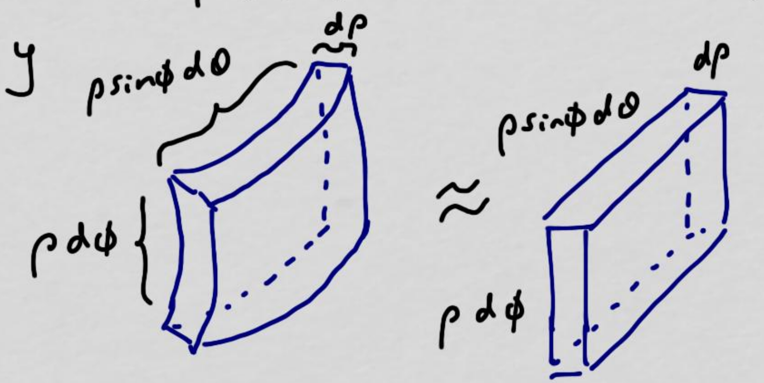
Spherical coordinates, as the name implies, is very good when integrating in spherical or sphere-like volumes

what is dV in spherical ($dV = dz\,dy\,dx$ in Cartesian, $dV = r\,dz\,dr\,d\theta$ in Cylindrical)



imagine a small piece of shell formed by $d\rho$, $d\phi$, $d\theta$ (small changes in ρ, ϕ, θ)

so, dV is the volume of



if $d\rho, d\theta, d\phi$ are very small

so, $dV \approx$ volume of rectangle $= (\rho\, d\phi)(\rho \sin\phi\, d\theta)(d\rho)$

$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

example

$$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} dz\,dy\,dx$$

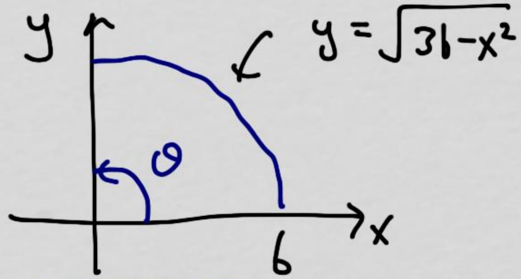
terrible integral in Cartesian

the upper bound of z: $z = \sqrt{72-x^2-y^2} \longrightarrow x^2+y^2+z^2 = 72$ is a sphere, so that is an indication to switch to spherical

the projection of the space onto xy-plane is

$0 \le x \le 6$

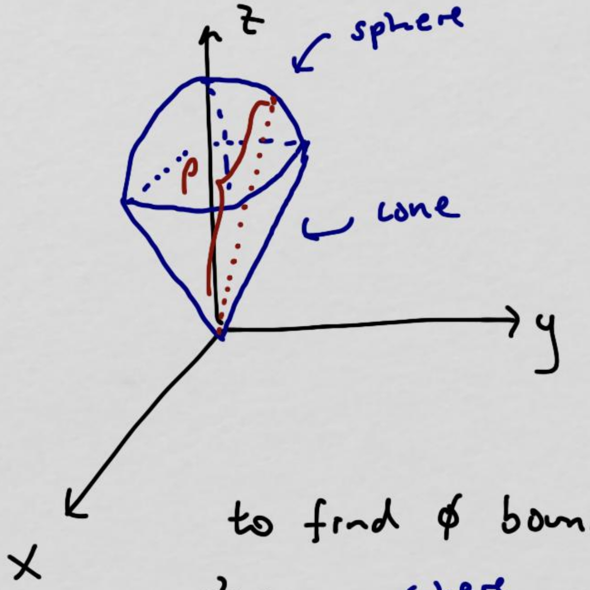
$0 \le y \le \underbrace{\sqrt{36-x^2}}_{\text{circle radius } 6}$



so, right away, we see $\boxed{0 \le \theta \le \pi/2}$

now we look at z bounds

$$\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{72-x^2-y^2}}_{\text{sphere radius } \sqrt{72}}$$

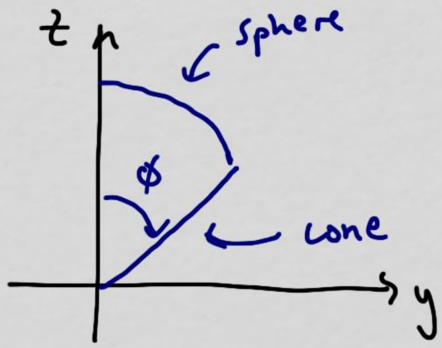


$0 \le \theta \le \pi/2$ so only a quarter of it

ρ bounds are: from origin ($\rho = 0$) to the surface formed by the sphere above ($\rho = \sqrt{72}$)

$$\boxed{0 \le \rho \le \sqrt{72}}$$

to find ϕ bounds, let's look at yz-trace



the cone on yz-plane is $z = y$ which has slope of 1 (45° or $\frac{\pi}{4}$)

ϕ starts at z-axis ($\phi = 0$) to the cone ($\phi = \pi/4$)

$$\boxed{0 \le \phi \le \frac{\pi}{4}}$$

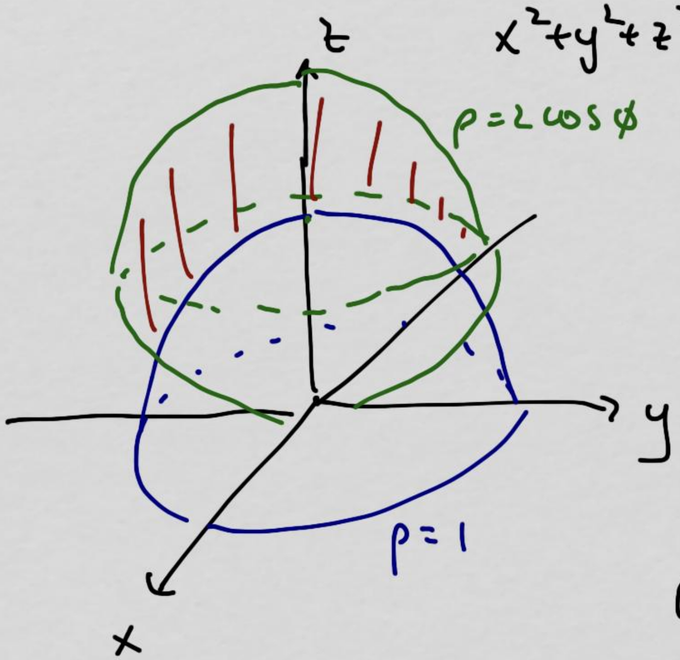
convert $$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} \underbrace{dz\,dy\,dx}_{dV}$$

to $$\underset{\theta}{\int_0^{\pi/2}} \underset{\phi}{\int_0^{\pi/4}} \underset{\rho}{\int_0^{\sqrt{72}}} \overbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta} = \cdots = \boxed{12\sqrt{72}\,\pi\left(1-\frac{1}{\sqrt{2}}\right)}$$

example Find volume of space outside $\rho = 1$ and inside $\rho = 2\cos\phi$

$\rho = 1 \rightarrow \sqrt{x^2+y^2+z^2} = 1 \rightarrow x^2+y^2+z^2 = 1$ sphere radius 1 center at origin

$$\rho = 2\cos\phi \rightarrow \sqrt{x^2+y^2+z^2} = \frac{2\overset{z}{\rho\cos\phi}}{\rho} = \frac{2z}{\sqrt{x^2+y^2+z^2}}$$



$x^2+y^2+z^2 = 2z \rightarrow x^2+y^2+(z-1)^2 = 1$

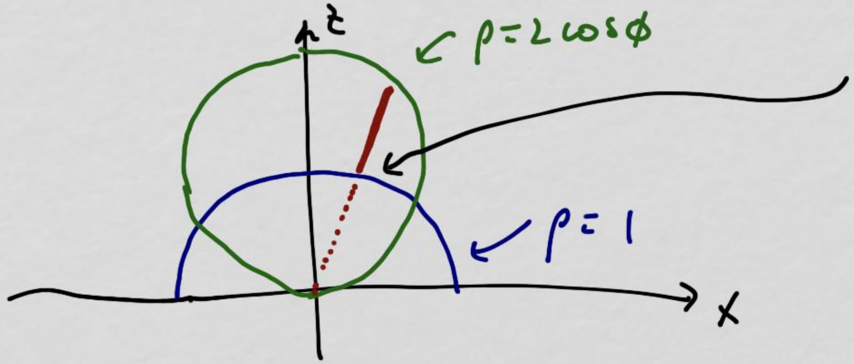
sphere radius 1 center at $(0,0,1)$

space between:  mushroom cap

θ bounds should be obvious: $\boxed{0 \le \theta \le 2\pi}$

(want all the way about z-axis)

ρ bounds aren't too bad: let's look at xz-trace

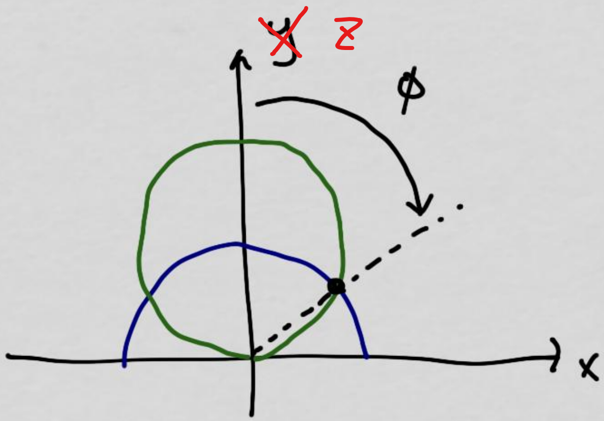


ρ starts at blue sphere ($\rho = 1$)

ends at green sphere ($\rho = 2\cos\phi$)

so, $\boxed{1 \le \rho \le 2\cos\phi}$

ϕ can also be found from the same trace



ϕ's upper bound is from line through intersection of the 2 circles

$$\left.\begin{array}{l}\rho = 2\cos\phi\\ \rho = 1\end{array}\right\} \quad 2\cos\phi = 1 \rightarrow \cos\phi = \frac{1}{2}$$

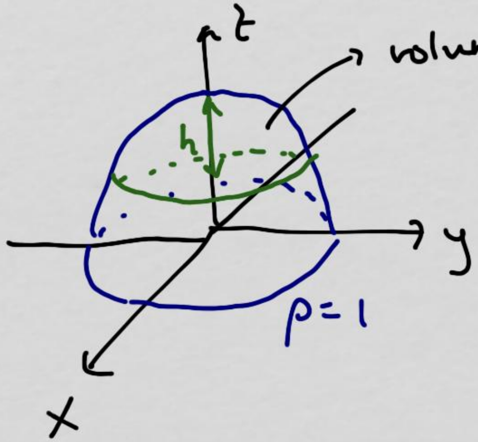
$$\phi = \frac{\pi}{3}$$

so, $\boxed{0 \le \phi \le \frac{\pi}{3}}$

volume : $\iiint_D dV = \int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos\phi} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} = \dots = \boxed{\frac{11\pi}{12}}$

(θ, ϕ, ρ)

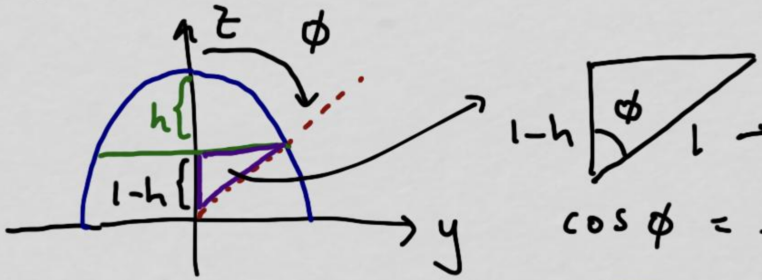
<u>example</u> Volume of spherical cap with height h cut from sphere of radius 1.



→ volume of this cap?

again, θ bounds are obvious: $\boxed{0 \le \theta \le 2\pi}$

ϕ is done like in last example

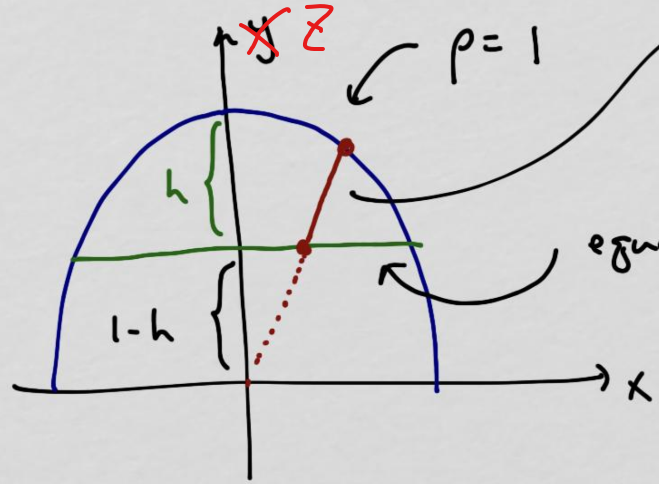


→ radius of sphere

$\cos\phi = \frac{1-h}{1} \rightarrow \phi = \cos^{-1}(1-h)$

$$\boxed{0 \le \phi \le \cos^{-1}(1-h)}$$

ρ is from the same trace



lower bound of ρ: plane $\rightarrow$ $\rho = (1-h)\sec\phi$

upper bound: 1

equation of this: $z = 1-h$

$$\downarrow$$

$$\rho\cos\phi$$

$$\rho\cos\phi = 1-h$$

$$\rho = \frac{1-h}{\cos\phi} = (1-h)\sec\phi$$

$$\boxed{(1-h)\sec\phi \le \rho \le 1}$$

volume of cap:

$$\boxed{\underset{\theta}{\int_0^{2\pi}} \underset{\phi}{\int_0^{\cos^{-1}(1-h)}} \underset{\rho}{\int_{(1-h)\sec\phi}^{1}} \overbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}^{dV}} = \dots = \frac{1}{3}\pi h^2 (3-h)$$