17.2 (part 2) Line Integrals of Vector Fields

the line integral of a path C through a scalar field f

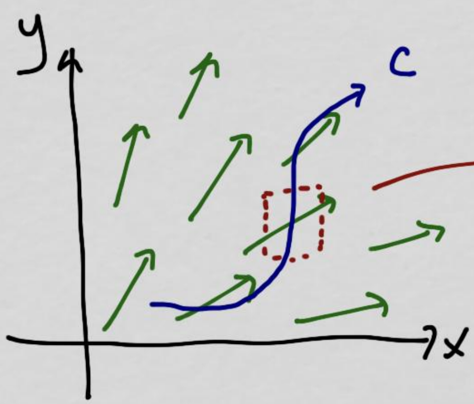
is $\int_C f\,ds$

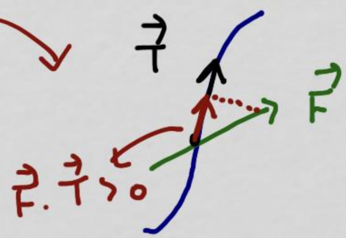
the line integral of a path C through a vector field $\vec{F}$

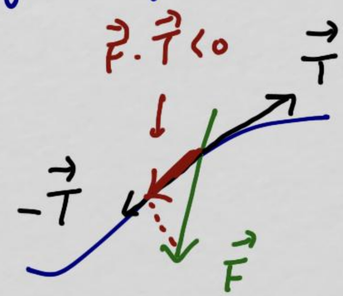
is defined as $\int_C \underbrace{\vec{F}\cdot\vec{T}}_{} \, ds$ $\vec{T}$: unit tangent vector on C

projection of $\vec{F}$ onto $\vec{T}$

in other words, $\vec{F}\cdot\vec{T}$ gives us the component of $\vec{F}$ along or against the direction of motion







$$\boxed{\int_C \vec{F}\cdot\vec{T}\,ds}$$ accumulates $\vec{F}\cdot\vec{T}$ along the entire path

if C is parametrized as $\vec{r}(t)$ $\quad a \le t \le b$

then $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$ and $ds = |\vec{r}\,'|\,dt$

so $$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\frac{\vec{r}\,'}{|\vec{r}\,'|}\,|\vec{r}\,'|\,dt = \boxed{\int_C \vec{F}\cdot\vec{r}\,'\,dt}$$

alternative form of $\int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{r}\,'dt$ can be written as $\dfrac{d\vec{r}}{dt}\,dt = d\vec{r}$

therefore, another form is $$\boxed{\int_C \vec{F}\cdot d\vec{r}}$$

common application is work : accumulation of the component of force along or against the direction of motion.

How to calculate $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$?

whatever form is used, the first step is to parametrize C

then find $\vec{T}$ or $\vec{r}\,'$ or $d\vec{r}$ and use one of the forms above

after rewriting $\vec{F}$ in terms of the parameter.

<u>example</u> $\vec{F} = \langle xy,\ y-x \rangle$

C: line segment from $(0,1)$ to $(2,4)$

parametrize C : $\vec{r}(t) = \langle 0, 1\rangle + t\,\langle 2, 3\rangle \quad 0 \le t \le 1$

$$= \langle 2t, 3t+1 \rangle \quad 0 \le t \le 1$$

just like with scalar line integral, the choice of parametrization does <u>NOT</u> affect the integral.

$$\int_C \vec F \cdot \vec T \, ds \qquad \vec r(t) = \langle \overset{x}{2t}, \overset{y}{3t+1} \rangle \qquad 0 \le t \le 1$$

$$\vec r\,' = \langle 2, 3 \rangle \qquad \vec F = \langle xy, y - x \rangle$$

$$|\vec r\,'| = \sqrt{13}$$

$$= \int_0^1 \overbrace{\langle \underset{x}{2t} \cdot \underset{y}{(3t+1)}, \underset{y}{3t+1} - \underset{x}{2t} \rangle}^{\vec F} \cdot \underbrace{\frac{\langle 2, 3 \rangle}{\sqrt{13}}}_{\vec T} \underbrace{\sqrt{13}\, dt}_{ds}$$

$$= \int_0^1 \underbrace{\langle 6t^2 + 2t, t+1 \rangle}_{\vec F} \cdot \underbrace{\langle 2, 3 \rangle}_{\vec r\,'} \, dt$$

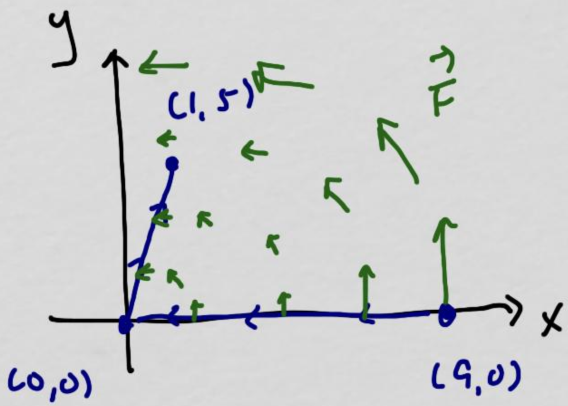
automatically becomes $\int_C \vec F \cdot \underbrace{\vec r\,' \, dt}_{d\vec r}$

$$= \int_0^1 (12t^2 + 7t + 3)\, dt = \cdots = \boxed{\frac{25}{2}}$$

this would be the work done if $\vec F$ is a force vector field in moving an object along C

example $\vec{F} = \langle -y, x \rangle$

C : line segment from (9, 0) to (0,0) then
line segment from (0,0) to (1, 5)



parametrize C

C_1: from (9,0) to (0,0)

$\vec{r}_1(t) = \langle 9, 0 \rangle + t \langle -9, 0 \rangle \quad 0 \leq t \leq 1$

$= \langle 9 - 9t, 0 \rangle \quad 0 \leq t \leq 1$

$\vec{r}_1{}' = \langle -9, 0 \rangle$

C_2: from (0,0) to (1, 5)

$\vec{r}_2 = \langle 0, 0 \rangle + t \langle 1, 5 \rangle \quad 0 \leq t \leq 1$

$= \langle t, 5t \rangle$

$\vec{r}_2{}' = \langle 1, 5 \rangle$

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$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt$$

$$= \underbrace{\int_0^1 \underbrace{\langle 0, 9-9t\rangle}_{\substack{\vec{F} \text{ using } x,y\\ \text{of } \vec{r}_1}} \cdot \underbrace{\langle -9, 0\rangle}_{\vec{r}_1\,'}\,dt}_{C_1} + \underbrace{\int_0^1 \underbrace{\langle -5t, t\rangle}_{\substack{\vec{F} \text{ using } x,y\\ \text{of } \vec{r}_2}} \cdot \underbrace{\langle 1, 5\rangle}_{\vec{r}_2\,'}\,dt}_{C_2}$$

$$= \int_0^1 0\,dt + \int_0^1 (-5t+5t)\,dt = \boxed{0}$$

this is 0 because $\vec{F}\cdot\vec{r}\,'$ is zero everywhere on C

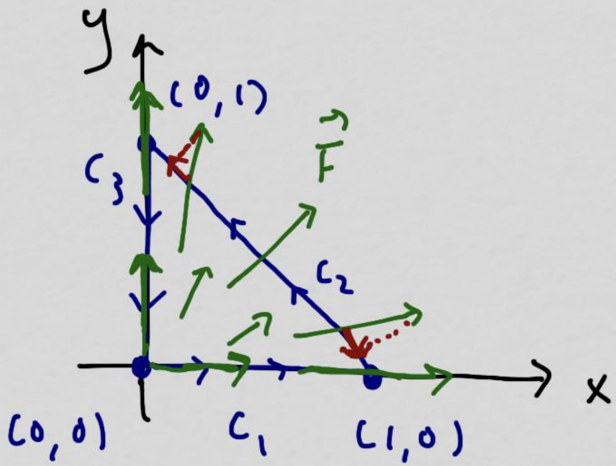
(vectors are always orthogonal to C, so no component is along / against motion) if C is a closed path (same starting and ending locations)

then $\int_C \vec{F}\cdot\vec{T}\, ds$ is also called the circulation of $\vec{F}$ on C

example $\vec{F} = \langle x, y \rangle$

C: line segment from $(0,0)$ to $(1,0)$
then line segment from $(1,0)$ to $(0,1)$
then line segment from $(0,1)$ to $(0,0)$

same starting and ending locations



note whatever $\vec{F}\cdot\vec{T}$ we accumulate on C_1 ($\vec{F}\cdot\vec{T} > 0$) will get cancelled out by accumulating $\vec{F}\cdot\vec{T}$ on C_3 ($\vec{F}\cdot\vec{T} < 0$)

on C_2, the accumulation of $\vec{F}\cdot\vec{T}$ on first half (with $\vec{F}\cdot\vec{T}$ against motion) gets cancelled out on the second half

the net accumulation on C_2 is zero because half of the path sees $\vec{F}$ with the motion and the half against note this happens when the vector field is symmetric with respect to a line through the origin of the path

verify.

$C_1: \vec{r}_1 = \langle t, 0 \rangle \quad 0 \le t \le 1$

$C_2: \vec{r}_2 = \langle 1-t, t \rangle \quad 0 \le t \le 1$

$C_3: \vec{r}_3 = \langle 0, 1-t \rangle \quad 0 \le t \le 1$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \vec{r}\,'\, ds$$

$$= \int_0^1 \underbrace{\langle t, 0 \rangle}_{\substack{\vec{F} \text{ w/} \\ x,y \text{ from } \vec{r}_1}} \cdot \underbrace{\langle 1, 0 \rangle}_{\vec{r}_1\,'} dt + \int_0^1 \underbrace{\langle 1-t, t \rangle}_{\substack{\vec{F} \text{ w/ } x,y \\ \text{from } \vec{r}_2}} \cdot \underbrace{\langle -1, 1 \rangle}_{\vec{r}_2\,'} dt + \int_0^1 \langle 0, 1-t \rangle \cdot \langle 0, -1 \rangle dt$$

same idea on $\vec{r}_3$

$$= \int_0^1 t\, dt + \int_0^1 (2t-1)\, dt + \int_0^1 (t-1)\, dt = \frac{1}{2} + (1-1) + \frac{1}{2} - 1 = \boxed{0}$$

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another form of $\int_C \vec{F} \cdot \vec{T}\, ds$

let $\vec{F} = \langle f, g \rangle$ and C parametrized by $\vec{r} = \langle x, y \rangle$ $a \le t \le b$

then $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|} = \dfrac{\langle x', y' \rangle}{\sqrt{(x')^2 + (y')^2}}$ $\quad ds = |\vec{r}\,'|\, dt = \sqrt{(x')^2 + (y')^2}\, dt$

$\int_C \vec{F} \cdot \vec{T}\, ds$ becomes $\displaystyle\int_C \langle f, g \rangle \cdot \frac{\langle x', y' \rangle}{\cancel{\sqrt{(x')^2 + (y')^2}}} \cancel{\sqrt{(x')^2 + (y')^2}}\, dt$

$$= \int_C f \underbrace{x'}_{\frac{dx}{dt}} dt + g \underbrace{y'}_{\frac{dy}{dt}} dt = \boxed{\int_C f\, dx + g\, dy}$$

components of $\vec{F}$

we can choose to evaluate that form by converting it back to $\int_C \vec{F}\cdot\vec{T}\,ds$ or stay with it using the parametrization we choose to use.

<u>example</u>

$$\int_C xy\,dx + (x+y)\,dy$$

$C : (0,0)$ to $(1,1)$ along $y = x^2$

one way to evaluate : convert back to $\int_C \vec{F}\cdot\vec{T}\,ds$

$$\int_C xy\,dx + (x+y)\,dy = \int_C \langle xy, x+y\rangle \cdot \langle dx, dy\rangle$$

$$= \int_C \langle xy, x+y\rangle \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle}_{\vec{r}\,'} dt$$

$C : \vec{r}(t) = \langle t, t^2\rangle \qquad 0 \le t \le 1$

$\vec{r}\,' = \langle 1, 2t\rangle$

$$\int_C \langle xy, x+y\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt \qquad r = \langle \underset{x}{t}, \underset{y}{t^2}\rangle \qquad 0 \le t \le 1$$

$$= \int_0^1 \langle t^3, t+t^2\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^1 (3t^3 + 2t^2)\,dt = \boxed{\frac{17}{12}}$$

another way, stay with $\int_C xy\,dx + (x+y)\,dy$

$$C: \vec{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2}\rangle \qquad 0 \le t \le 1$$

$x = t$, then $\frac{dx}{dt} = 1$ then $dx = dt$

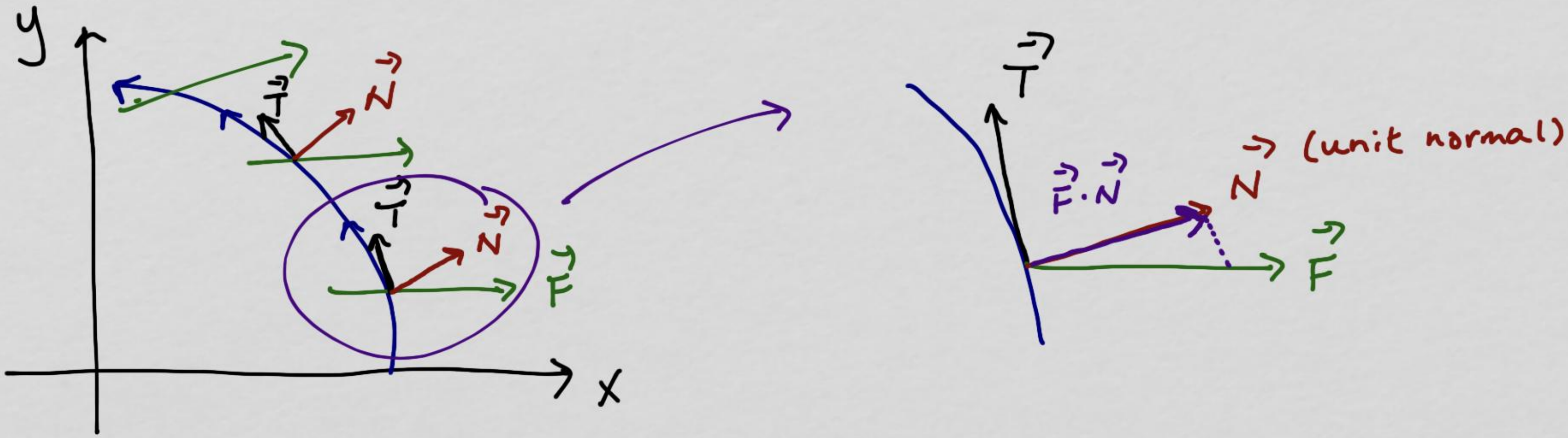
$y = t^2$, then $\frac{dy}{dt} = 2t$ then $dy = 2t\,dt$

$$\int_0^1 \underset{xy}{t^3}\ \underset{dx}{dt} + \underbrace{(t+t^2)}_{x+y}\underbrace{(2t)\,dt}_{dy} = \int_0^1 (t^3 + 2t^2 + 2t^3)\,dt = \boxed{\frac{17}{12}}$$

same

If we use the normal vector instead of the tangent vector in

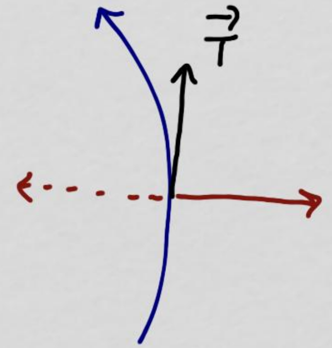
$\int_C \vec{F} \cdot \vec{T} \, ds$, then we end with the <u>flux integral</u>



the flux integral $\int_C \vec{F} \cdot \vec{N} \, ds$ accumulates the component normal to the path (the component that does not contribute to work)

if $\vec{F}$ is the flow of fluid, then $\int_C \vec{F} \cdot \vec{N} \, ds$ gives us the amount of fluid that flows through a barrier in the shape of C

note there are two normal vectors: one to the right of $\vec{T}$ and one to the left



both red vectors are normal to path

$\boxed{\text{unless told otherwise, we choose the }\textit{unit}\text{ normal vector to the RIGHT of } \vec{T}}$

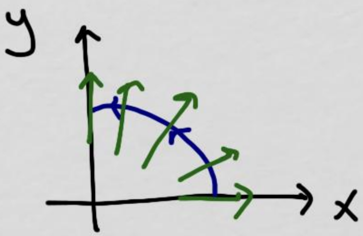
$\vec{T}$ cannot change length, so $\vec{T}'$ is always a result of the turning on the path C and $\vec{T}'$ is always normal to $\vec{T}$

then $\dfrac{\vec{T}'}{|\vec{T}'|}$ is a unit vector normal to $\vec{T}$

$$\boxed{\vec{N} : \frac{\vec{T}'}{|\vec{T}'|} \text{ or } -\frac{\vec{T}'}{|\vec{T}'|} \quad \text{whichever is to the } \underline{\underline{\text{right}}} \text{ of } \vec{T}}$$

<u>example</u> $\vec{F} = \langle x, y \rangle$

C: $(1,0)$ to $(0,1)$ along $x^2+y^2=1$



C: $\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le \pi/2$

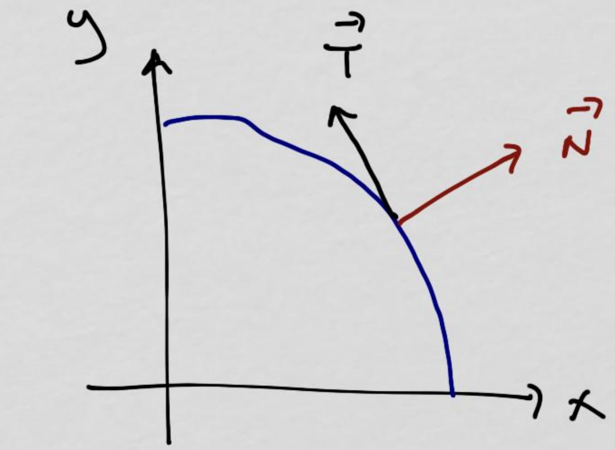
$\vec{r}\,' = \langle -\sin t, \cos t \rangle \qquad |\vec{r}\,'| = 1$

$\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|} = \langle -\sin t, \cos t \rangle$

$\vec{T}\,' = \langle -\cos t, -\sin t \rangle \qquad |\vec{T}\,'| = 1$

so, we choose $\dfrac{\vec{T}\,'}{|\vec{T}\,'|} = \langle -\cos t, -\sin t \rangle$ or $-\dfrac{\vec{T}\,'}{|\vec{T}\,'|} = \langle \cos t, \sin t \rangle$

whichever is to the right of $\vec{T}$



note if $\vec{N}$ is to the right of $\vec{T}$

then both components of $\vec{N}$ must

be nonnegative for $0 \le t \le \pi/2$

so, we choose $\vec{N} = \langle \cos t, \sin t \rangle$

$$\int_C \vec{F} \cdot \vec{N}\, ds = \int_0^{\pi/2} \underbrace{\langle \cos t, \sin t \rangle}_{\substack{\vec{F} = \langle x, y \rangle \\ \text{using } x, y \\ \text{of } \vec{r}}} \cdot \underbrace{\langle \cos t, \sin t \rangle}_{\vec{N}}\, dt$$

$$ds = |\vec{r}\,'|\, dt = (1)\, dt$$

$$= \int_0^{\pi/2} \big(\underbrace{\cos^2 t + \sin^2 t}_{1}\big)\, dt = \boxed{\frac{\pi}{2}}$$