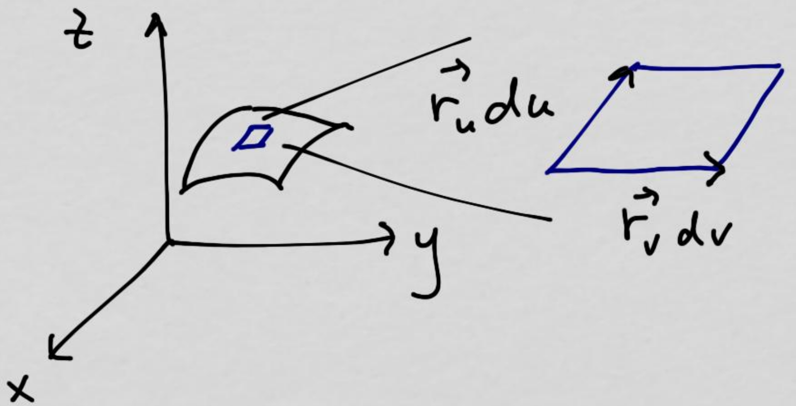
17.6 Surface Integrals (part 3)

last time: surface integrals of functions

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$$= \iint_R f(x,y,z)\sqrt{1+z_x^2+z_y^2}\,dA$$

$dS = |\vec{r}_u \times \vec{r}_v|\,dA$ where $\vec{r}_u \times \vec{r}_v$ is the normal vector of the surface



there are actually two normal vectors: $\vec{r}_u \times \vec{r}_v$ and $\vec{r}_v \times \vec{r}_u$

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in $\iint_S f(x,y,z)\,dS = \iint_R f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,dA$, we didn't care which of the two to use, because we used the magnitude in the integral, and the scalar field $f(x,y,z)$ has no directions.

But, when integrating a vector field, the direction matters.

<u>oriented surface</u>

we need to define an orientation for the surface S by choosing which of the two normal vectors to use

by convention, we normally choose the <u>upward-pointing</u> normal vector

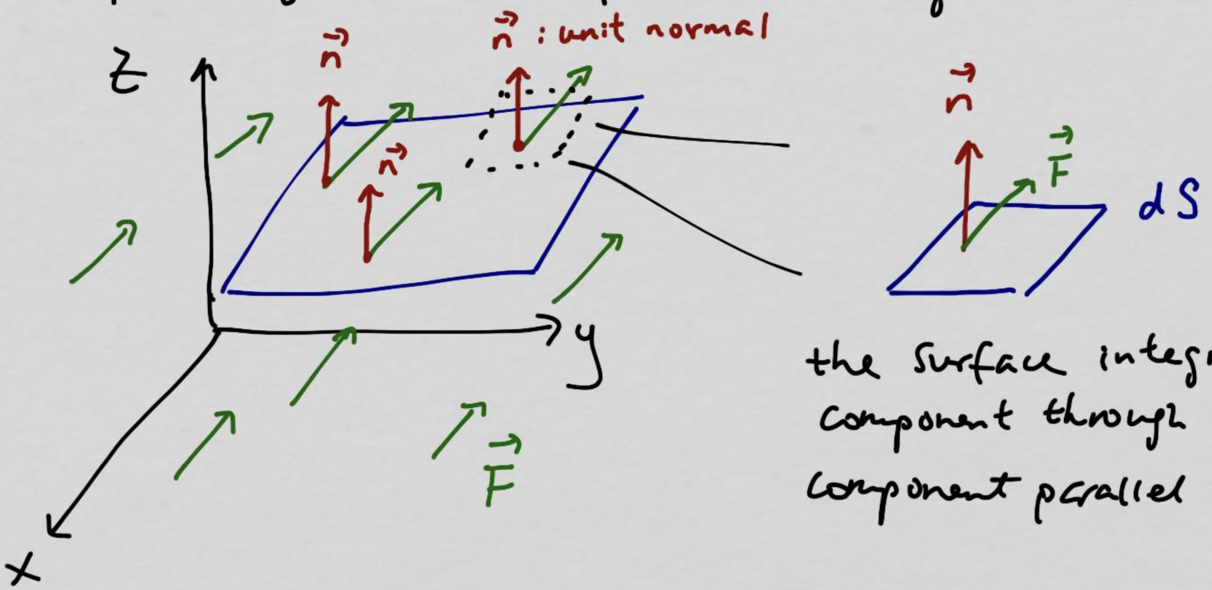
→ we call the surface <u>positively-oriented</u>

(if the surface is closed, for example a sphere, we normally choose the <u>outward-pointing</u> normal vector to be positively oriented) we can choose other orientations (for example, pointing in the positive x-axis) if we want to, but we need to state it.

if the orientation is not specified, assume we want the upward/outward normal.

a surface integral of a vector field is often called a <u>flux integral</u>

it measures the amount of "flow" of the vector field <u>through</u> the surface (just like the flux line integral we saw earlier)



the surface integral accumulates the component through the surface → component parallel to $\vec{n}$ (dot product)

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So, we accumulate $\vec{F}\cdot\vec{n}\,dS$ all over S

write:
$$\iint_S \vec{F}\cdot \underbrace{d\vec{S}}_{\text{oriented surface}} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

unit normal

the usual $dS = |\vec{r}_u \times \vec{r}_v|\,dA$

So, it turns into

$$\iint_S \vec{F}\cdot \underbrace{\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}}_{\vec{n}} \underbrace{|\vec{r}_u\times\vec{r}_v|\,dA}_{dS} = \boxed{\iint_R \vec{F}\cdot\underbrace{(\vec{r}_u\times\vec{r}_v)}\,dA}$$

or $\vec{r}_v\times\vec{r}_u$ whichever is positively oriented (usually up or out)

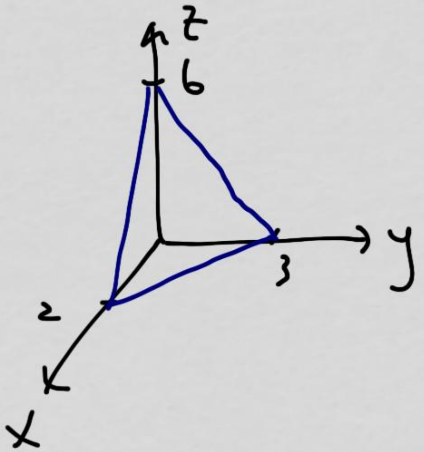
where u, v can be

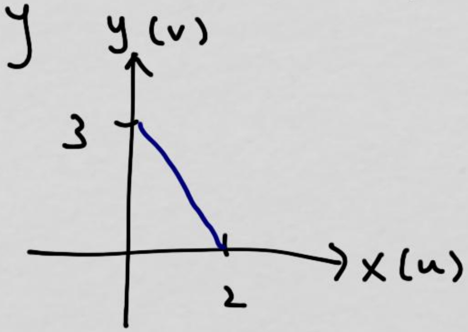
if the surface is explicitly defined as $z = z(x,y)$, then the alternative form is $\iint_R (-f z_x - g z_y + h)\, dA$ where $\vec{F} = \langle f, g, h \rangle$

and $\langle -z_x, -z_y, 1 \rangle$ is the positively oriented normal

<u>example</u> $\vec{F} = \langle x, y, z \rangle$ S: plane $3x + 2y + z = 6$ in the first octant

normal vector points upward.





parametrize S:

let $u = x$, $v = y$, $z = 6 - 3x - 2y = 6 - 3u - 2v$

$0 \le u \le 2$, $0 \le v \le 3 - \frac{3}{2}u$

$\vec{r}(u,v) = \langle u, v, 6 - 3u - 2v \rangle$

$$\vec{r}_u = \langle 1, 0, -3 \rangle$$

$$\vec{r}_v = \langle 0, 1, -2 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$$

important: is this an upward-pointing normal?
(because that is the "positive" orientation)

yes, because z-component is positive

if not, reverse the cross product

$$\vec{F} = \langle x, y, z \rangle \qquad \vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$$

$$\vec{F} = \langle u, v, 6-3u-2v \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

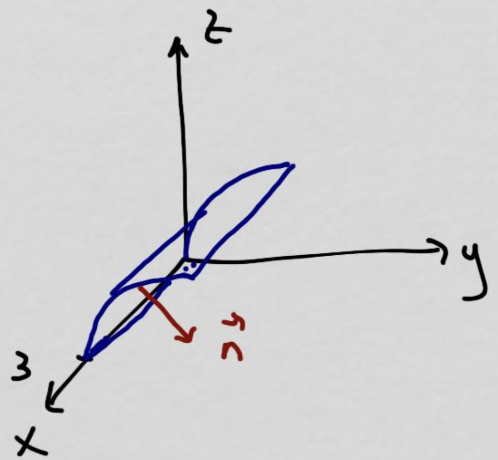
$$= \underset{u}{\int_0^2} \underset{v}{\int_0^{3-\frac{3}{2}u}} \underbrace{\langle u, v, 6-3u-2v \rangle}_{\vec{F}} \cdot \underbrace{\langle 3, 2, 1 \rangle}_{\vec{r}_u \times \vec{r}_v} \underbrace{dv\, du}_{dA}$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}u} 6 \, dv\, du = \cdots = \boxed{18}$$

(some measure of the net flow through the surface in the same direction as the normal vector)

example $\vec{F} = \langle -y, x, 1 \rangle$ S: cylinder $y = z^2$, $0 \le x \le 3$, $0 \le z \le 1$

normal points toward positive y-axis



parametrize S: let $u = x$, $v = z$, $y = v^2$

$0 \le u \le 3$, $0 \le v \le 1$

$$\vec{r}(u,v) = \langle u, v^2, v \rangle$$

$$\vec{r}_u = \langle 1, 0, 0 \rangle$$

$$\vec{r}_v = \langle 0, 2v, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0, -1, 2v \rangle$$

does this point toward positive y-axis?

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no, because the $\vec{j}$ - component is negative

so, reverse the cross product

use $\vec{r}_v \times \vec{r}_u = \langle 0, 1, -2v \rangle$ as the normal vector

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot \underbrace{(\vec{r}_v \times \vec{r}_u)}_{\text{this is "positive" this time}} dA$$

$\vec{F} = \langle -y, x, 1 \rangle$

$\vec{r}(u,v) = \langle u, v^2, v \rangle$

$$= \int_{0}^{3} \int_{0}^{1} \underbrace{\langle -v^2, u, 1 \rangle}_{\vec{F}} \cdot \underbrace{\langle 0, 1, -2v \rangle}_{\vec{r}_v \times \vec{r}_u} \, dv\, du$$

$u \quad v$

$$= \int_0^3 \int_0^1 (u - 2v)\, dv\, du = \cdots = \boxed{\frac{3}{2}}$$

example $\vec{F} = \dfrac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$ S: sphere of radius a, normal outward

parametrize S: use spherical coordinates

let $u=\phi$, $v=\theta$ $\quad 0 \le u \le \pi$, $\quad 0 \le v \le 2\pi$

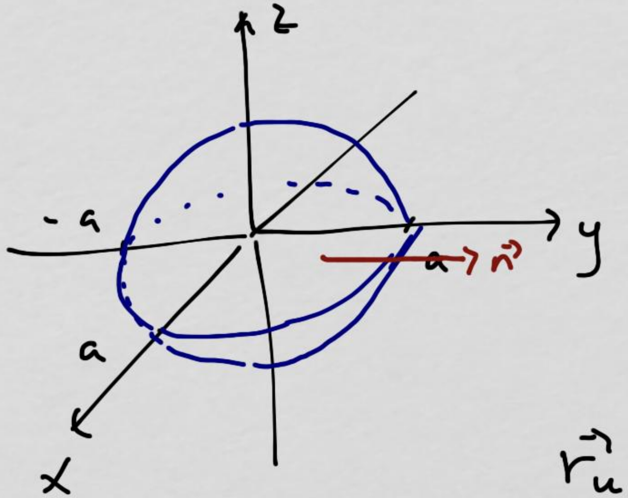
$$\vec{r}(u,v) = \langle \underbrace{a\sin u\cos v}_{\rho\sin\phi\cos\theta}, \underbrace{a\sin u\sin v}_{\rho\sin\phi\sin\theta}, \underbrace{a\cos u}_{\rho\cos\phi} \rangle$$

$$\vec{r}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u\rangle$$

$$\vec{r}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$$

does this point outward?



note in the first octant $(x>0,\ y>0,\ z>0)$

we want $\vec{n}$ to have positive components

in first octant, $0 \le u \le \pi/2$, $0 \le v \le \pi/2$

ϕ θ

$$\vec{r}_u \times \vec{r}_v = \langle \underbrace{a^2 \sin^2 u}_{\ge 0} \cos v,\ \underbrace{a^2 \sin^2 u}_{\ge 0} \sin v,\ \underbrace{a^2}_{\ge 0} \cos u \sin u \rangle$$

do they have the right sign?

yes, because $\cos v$, $\sin v$, $\cos u$, $\sin u$ all are positive in first octant

so, $\vec{r}(u,v)$ appears to have the right direction

(if unsure, check other octants

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

$$\vec{F} = \frac{-\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

$$\vec{F}(u,v) = \langle \overset{x}{a\sin u\cos v}, \overset{y}{a\sin u\sin v}, \overset{z}{a\cos u}\rangle$$

$$= \int_{\underset{v}{0}}^{2\pi}\int_{\underset{u}{0}}^{\pi} \underbrace{\frac{-\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle}{a^3}}_{\vec{F}} \cdot \underbrace{\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\cos u\sin u\rangle}_{\vec{r}_u\times\vec{r}_v}\ \underbrace{du\,dv}_{dA}$$

$$= \int_0^{2\pi}\int_0^{\pi} -(\sin^3 u\cos^2 v + \sin^3 u\sin^2 v + \cos^2 u\sin u\rangle\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi} -\sin u\,du\,dv = \cdots = \boxed{-4\pi}$$

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the flux is negative because $\vec{F}$ points toward the origin while the normal vector points outward (flow and normal are in opposite directions)

the negative sign tells us the net flow is opposite to normal vector (inward)

notice a disappeared. This is only true if $\vec{F} = \dfrac{\pm\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} = \dfrac{\pm\vec{r}}{|\vec{r}|^3}$

if the power of $|\vec{r}|$ were different, the flux would depend on a.

<u>example</u> $\vec{F} = -\langle x, y, z\rangle$ S: same sphere

$\vec{r}(u,v)$: same

$\vdots$

$\vec{r}_u \times \vec{r}_v = \langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,$
$a^2 \cos u \sin u\rangle$

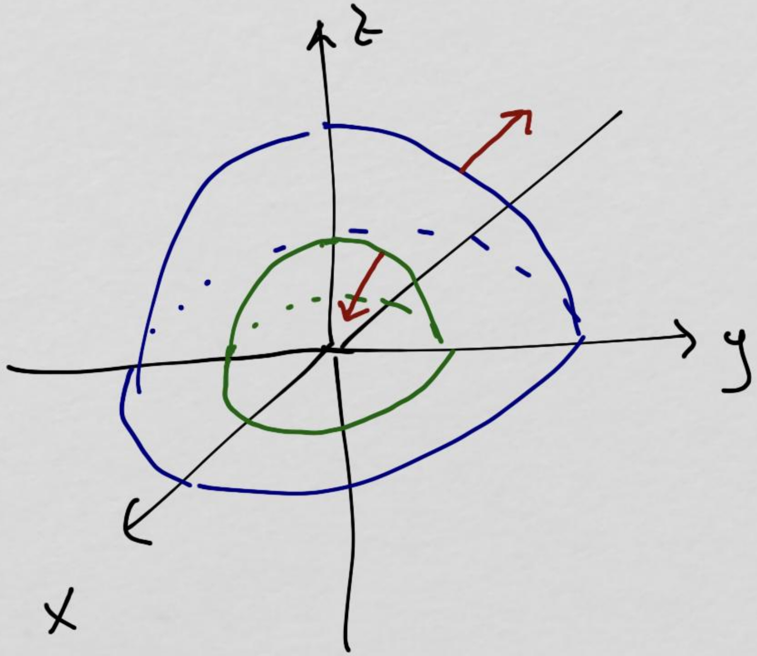
$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \underbrace{-\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle}_{\vec{F}}$$

$$\cdot \langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\cos u\sin u\rangle\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi} -\left(a^3\sin^3 u\cos^2 v + a^3\sin^3 u\sin^2 v + a^3\cos^2 u\sin u\right) du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi} -a^3\sin u\, du\, dv = \boxed{-4\pi a^3}$$

if we have two spheres, one inside another, and S consists of both surfaces, then <u>outward normal</u> means the outside sphere's normal points <u>out</u> while the inside sphere's normal points <u>in</u>



because we want the normal to be the way out of the surface (in close to inside, the way out is toward origin)