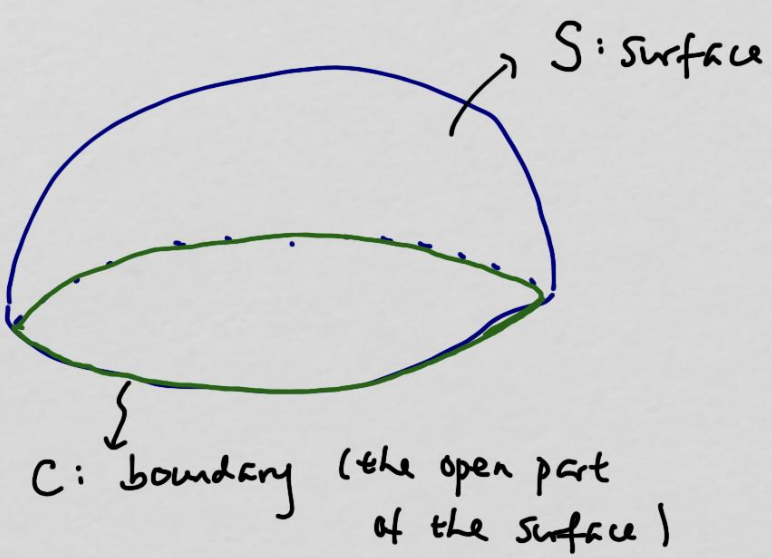
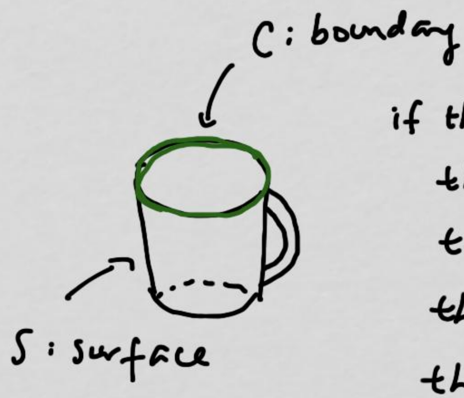
17.7 Stokes' Theorem (part 1)

Stokes' Theorem relates a surface integral of the curl of the vector field to a line integral along the boundary of the surface.



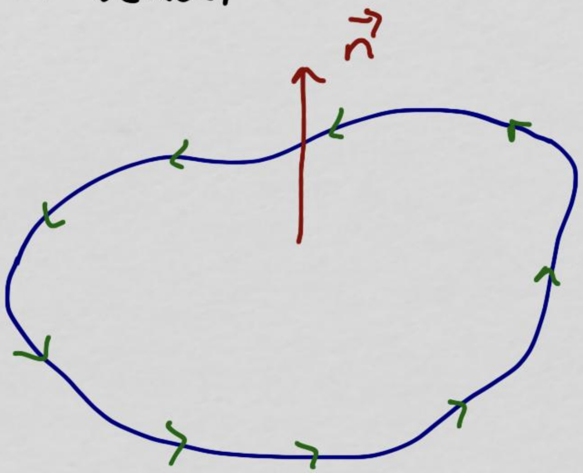


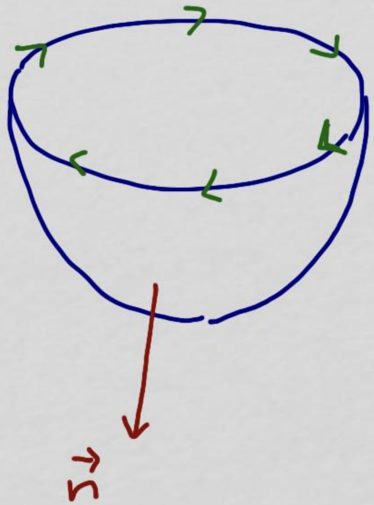
if the surface is the surface of the mug, then the bounday is the open part

both the surface and the boundary curve are <u>oriented</u>

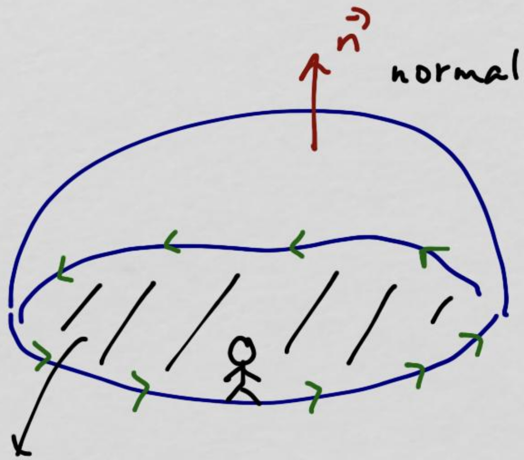
(surface orientation determined by the normal vector, curve orientation is determined by the direction of increasing "t") the direction of the surface's normal vector and the curve's orientation in Stokes' Theorem are related by the <u>right-hand rule</u> :

if the normal vector of the surface is along the thumb of your right hand, then the direction of increasing t on the curve is in the direction the rest of your fingers (on the right hand) natural bend.





another way to visualize the relationship: imagine you are walking along the boundary curve in the direction of increasing t, then the region enclosed by the curve is on your left side and the normal vector points in the direction your head points



normal points up because your head points up.

region enclosed is on your left

for Stokes' Theorem to work correctly, the orientations of the surface and its boundary curve need to be as described above.

Stokes' Theorem

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

S: surface

C: boundary curve

where $d\vec{S} = \vec{n}\,dS = (\vec{r}_u \times \vec{r}_v)\,du\,dv$

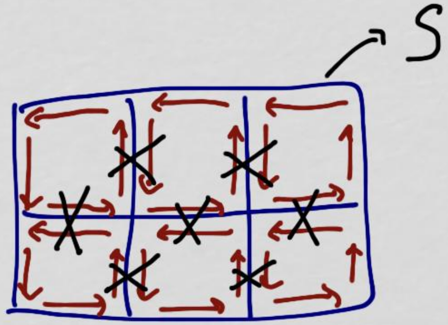
or $\underbrace{(\vec{r}_v \times \vec{r}_u)}_{\text{whichever is the "positive" direction}}$

the theorem is named in honor of Sir George Stokes but Sir William Thomson (aka Baron Kelvin) also made significant contributions.

why is $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$?

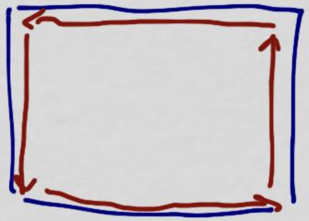
the explanation is the same as for Green's Theorem

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ is accumulation of curl of vector field on the surface



interior (adjacent) flows cancel

so it ends up looking like



which is the same as accumulation of vector field along the boundary $\oint_C \vec{F}\cdot d\vec{r}$

in fact, Green's Theorem is a special case of Stokes' Theorem

→ when the vector field is 2D and on xy-plane

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

if $\vec{F}$ is 2D and surface is 2D on xy-plane, then $\vec{F} = \langle f, g \rangle$

and $\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle$ and $\vec{n} = \vec{k}$

so,
$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_R \langle 0, 0, g_x - f_y\rangle \cdot \vec{k}\, dA$$
$$= \iint_R (g_x - f_y)\, dA$$

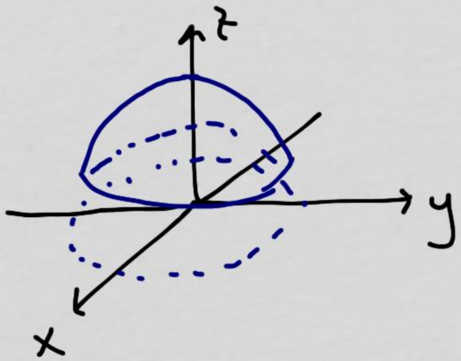
and we end up with
$$\iint_R (g_x - f_y)\, dA = \oint_C \vec{F}\cdot d\vec{r}$$
$$= \oint_C \langle f, g\rangle \cdot \langle dx, dy \rangle$$
$$= \oint_C f\,dx + g\,dy$$
(Green's Theorem)

example $\vec{F} = \langle y, -x, 0 \rangle$

S : sphere of radius 3, $z \geq 3/2$, normal is outward

let's verify Stokes' Theorem: $\displaystyle\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

left side $\left(\displaystyle\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} \right)$:



parametrize S

it's a sphere, so spherical coordinates seem to be a natural choice

let $u = \phi$, $v = \theta$

$$\boxed{\vec{r}(u,v) = \langle 3\sin u \cos v,\ 3\sin u \sin v,\ 3\cos u \rangle}$$

bounds: $\boxed{0 \leq v \leq 2\pi}$ is obvious

we want $z \geq 3/2$

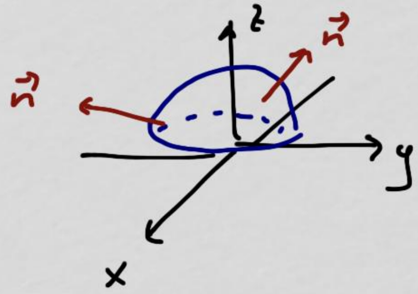
$3\cos u \geq 3/2$

$\cos u \geq 1/2 \rightarrow \boxed{0 \leq u \leq \pi/3}$

$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u \rangle$

$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\sin u\cos u \rangle$



is this vector "outward" as specified in the problem?

from the picture, for the vector to be outward, the

z-component ≥ 0

$9\sin u\cos u \geq 0$

$$\underbrace{0 \leq u \leq \pi/3, \qquad 0 \leq v \leq 2\pi}$$

notice $\sin u$ and $\cos u$ are ≥ 0

so, z-component $= 9\sin u\cos u \geq 0$

and the normal $\vec{r}_u \times \vec{r}_v$ $\underline{\text{is}}$ outward

(if not, do $\vec{r}_v \times \vec{r}_u$ instead)

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now perform the surface integral

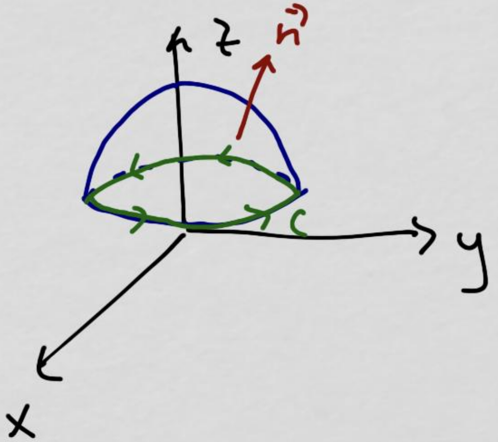
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} \qquad \vec{F} = \langle y, -x, 0 \rangle \quad \operatorname{curl} \vec{F} = \cdots = \langle 0, 0, -2 \rangle$$

$$= \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = \iint_R \operatorname{curl} \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA$$

$$= \int_{\underset{v}{0}}^{2\pi} \int_{\underset{u}{0}}^{\pi/3} \underbrace{\langle 0, 0, -2 \rangle}_{\operatorname{curl} \vec{F}} \cdot \underbrace{\langle 9\sin^2 u \cos v,\ 9\sin^2 u \sin v,\ 9 \sin u \cos u \rangle}_{\vec{r_u} \times \vec{r_v}} \underbrace{du\, dv}_{dA}$$

$$= \int_0^{2\pi} \int_0^{\pi/3} -18 \sin u \cos u \, du\, dv = \cdots = \boxed{-\frac{27\pi}{2}}$$

now let's do the line integral on the boundary $\oint_C \vec{F} \cdot d\vec{r}$



the boundary curve must be oriented counterclockwise when viewed from above

$S: \; x^2+y^2+z^2=9$

boundary curve is at $z = 3/2$

$$x^2+y^2+\left(\frac{3}{2}\right)^2 = 9 \rightarrow x^2+y^2 = \frac{27}{4}$$

so, C is a circle of radius $\frac{\sqrt{27}}{2}$

parametrize: $\vec{r}(t) = \left\langle \frac{\sqrt{27}}{2}\cos t, \; \frac{\sqrt{27}}{2}\sin t, \; \frac{3}{2} \right\rangle \quad 0 \le t \le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{\left\langle \frac{\sqrt{27}}{2}\sin t, \; -\frac{\sqrt{27}}{2}\cos t \right\rangle}_{\vec{F}} \cdot \underbrace{\left\langle -\frac{\sqrt{27}}{2}\sin t, \; \frac{\sqrt{27}}{2}\cos t \right\rangle}_{d\vec{r}} dt$$

(counterclockwise when viewed from above? yes.)

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$$= \int_0^{2\pi} \left(-\frac{27}{4}\sin^2 t - \frac{27}{4}\cos^2 t\right) dt = \int_0^{2\pi} -\frac{27}{4}\, dt = -\frac{27}{4}\cdot 2\pi = \boxed{-\frac{27\pi}{2}}$$

Same.

easier than surface integral? Maybe.

But the point is not really one being easier than the other

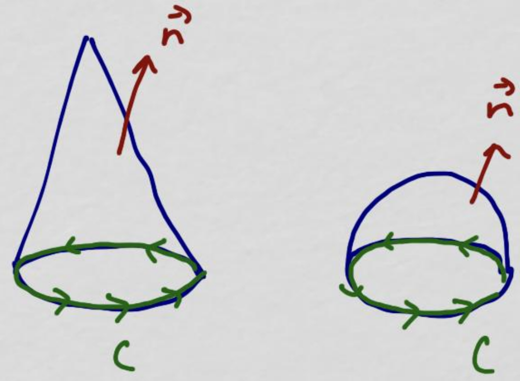
the point is that we have a choice

Stokes' Theorem: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

it tells us that if two surfaces have the same boundary C, then

the value of $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ will be the same because

$\oint_C \vec{F}\cdot d\vec{r}$ will be the same

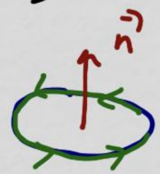


different surfaces but

the same boundary C

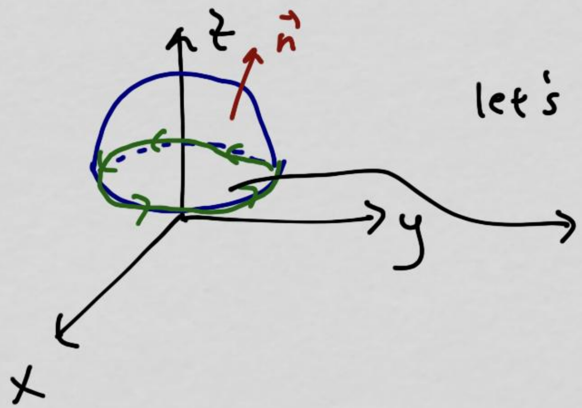
so, same $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

and would also be the same for

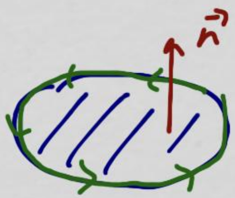


(flat surface w/ same boundary)

so, in the previous example, we could have chosen another surface w/ the same boundary to perform the surface integral



let's use the circle at $z = 3/2$ as our S



parametrize: (cylindrical) $u=r$, $v=\theta$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ 3/2 \rangle$$

$$0 \le u \le \frac{\sqrt{27}}{2} \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 0 \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \cdots = \langle 0, 0, u \rangle$$

out/upward?

yes, since $0 \le u \le \frac{\sqrt{27}}{2}$

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} \qquad \operatorname{curl}\vec{F} = \vec{\nabla} \times \langle y, -x, 0 \rangle = \cdots = \langle 0, 0, -2 \rangle$$

$$\int_0^{2\pi} \int_0^{\sqrt{27}/2} \langle 0,0,-2 \rangle \cdot \langle 0,0,u \rangle \, du\, dv$$

$$= \int_0^{2\pi} \int_0^{\sqrt{27}/2} -2u \, du\, dv = \int_0^{2\pi} -u^2 \Big|_0^{\sqrt{27}/2} dv = 2\pi \left(-\frac{27}{4} \right) = \boxed{-\frac{27\pi}{2}}$$

Stokes' can only be used if we are performing a surface
integral of the curl of the vector field
and if the surface is open (has a boundary)