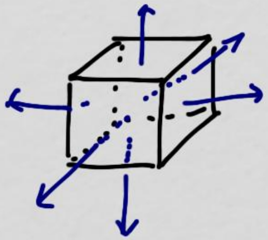
17.8 The Divergence Theorem (part 2)

Divergence Theorem: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$



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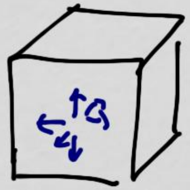

flux through surface

accumulation of divergence inside volume

what happens if we have a hollow region, for example, a cube with a hollow smaller cube inside?

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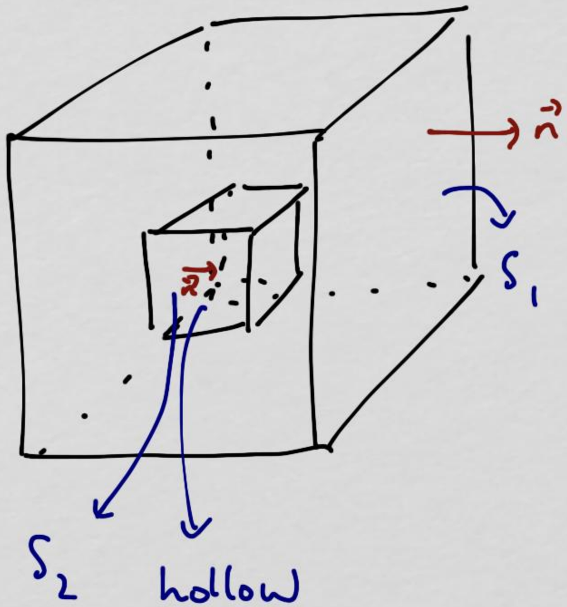

S : surface of the region between the cubes

S_1 : surface of the outer cube

S_2 : surface of the inner cube

D : space between the cubes

as usual, we want the normal vector to be outward

↳ away from the enclose region D

on S_1, $\vec{n}$ points away from center

on S_2, $\vec{n}$ points toward the center

(otherwise it would point into D, the region between the cubes, and that would not be pointing outward with respect to D)

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So, now we can modify the Divergence Theorem to calculate the outward flux through S

$$\underbrace{\iint_{S_2} \vec{F}\cdot\vec{n}\,dS}_{\text{outer surface}} + \underbrace{\iint_{S_1} -\vec{F}\cdot\vec{n}\,dS}_{\text{inner surface}} = \iiint_D \operatorname{div}\vec{F}\,dV$$

inner surface negative because the direction of the normal (w/o the outer one, "outward" is away from center, but now it needs to be the opposite of that)

$D \downarrow$ region/space between the inner and outer surfaces the left side can be written in terms of the divergence over the respective regions

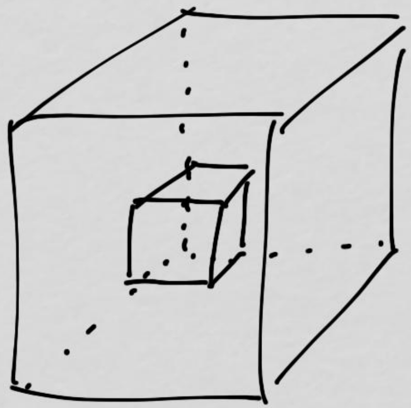
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_{D_1} \operatorname{div}\vec{F}\,dS - \iiint_{D_2} \operatorname{div}\vec{F}\,dS$$

D_1: region enclosed by S_1 (ignoring the presence of S_2)

D_2: region enclosed by S_2



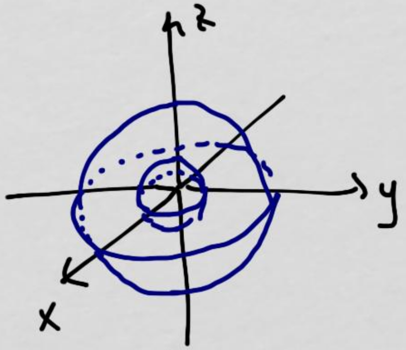
so, in the case with cubes, we do

$$\iiint_{\substack{\text{big}\\\text{cube}}} \operatorname{div}\vec{F}\,dV - \iiint_{\substack{\text{small}\\\text{cube}}} \operatorname{div}\vec{F}\,dV$$

$$\left(\text{or } \iiint_{\substack{\text{space}\\\text{between}\\\text{cubes}}} \operatorname{div}\vec{F}\,dV\right)$$

example $\vec{F} = \langle x, y, z \rangle$

D : between two spheres of radii 2 and 1, normal is outward



$D: \quad 1 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

so, the flux is $\displaystyle\iiint_D \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \underbrace{3}_{\operatorname{div}\vec{F}} \underbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}_{dV}$

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\Big|_1^2 \sin\phi \, d\phi\, d\theta$$

$$= 7\cdot 2\pi \int_0^{\pi} \sin\phi \, d\theta = 14\pi\left(-\cos\phi\right)\Big|_0^{\pi} = \boxed{28\pi}$$

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or, we can do $\iiint_{D_1} \text{div}\,\vec{F}\,dV - \iiint_{D_2} \text{div}\,\vec{F}\,dV$

D_1: space inside big sphere, ignoring the small one

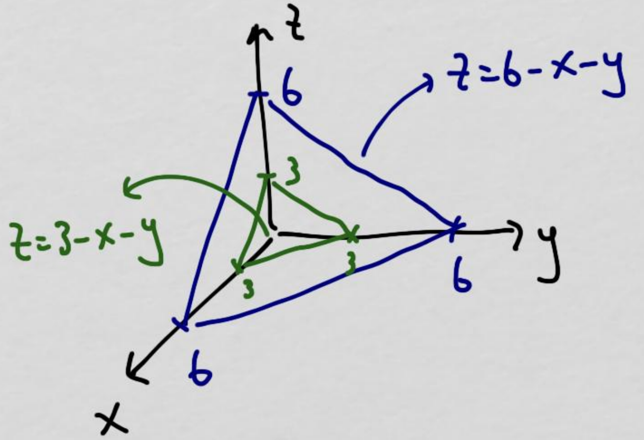
D_2: space inside the small sphere only

$$= 3\underbrace{\iiint_{D_1} dV}_{\text{volume of big sphere (radius 2)}} - 3\underbrace{\iiint_{D_2} dV}_{\text{volume of small sphere (radius 1)}} = 3\cdot\frac{4}{3}\pi(2)^3 - 3\cdot\frac{4}{3}\pi(1)^3$$

$$= 32\pi - 4\pi = \boxed{28\pi}$$

<u>example</u> $\vec{F} = \langle x^2 - y^2, z^2 \rangle$

D : region bounded by $z = 6 - x - y$ and $z = 3 - x - y$ in first octant



we can do this in 3 ways

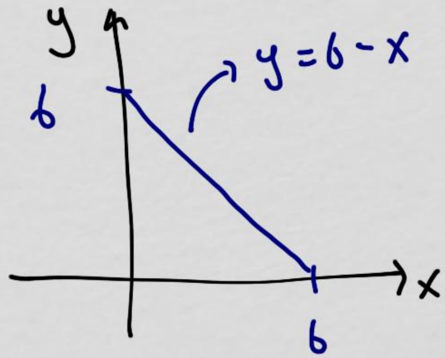
1) as surface integral
→ flux integral over 5 surfaces
(the two planes and the 3 coordinate planes)

2) $\iiint_D \operatorname{div} \vec{F}\, dV$ (but D is a bit messy to set up)

3) $\iiint_{\substack{\text{big}\\\text{region}}} \operatorname{div} \vec{F}\, dV - \iiint_{\substack{\text{small}\\\text{region}}} \operatorname{div} \vec{F}\, dV$

easiest in this case

big region

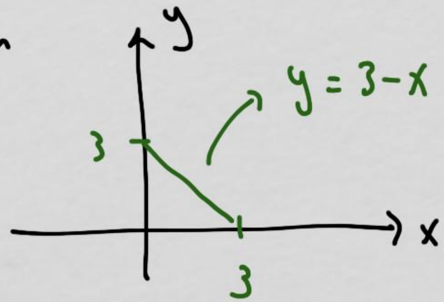


$0 \le x \le 6$

$0 \le y \le 6 - x$

$0 \le z \le 6 - x - y$

Small region



$0 \le x \le 3$

$0 \le y \le 3 - x$

$0 \le z \le 3 - x - y$

$\operatorname{div}\vec{F} = 2x - 2y + 2z = 2(x - y + z)$

so, the net outward flux is

$$\int_0^6 \int_0^{6-x} \int_0^{6-x-y} 2(x-y+z)\,dz\,dy\,dx - \int_0^3 \int_0^{3-x} \int_0^{3-x-y} 2(x-y+z)\,dz\,dy\,dx$$

$$= \dots = 108 - \frac{27}{4} = \boxed{\frac{405}{4}}$$

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<u>example</u> $\vec{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$

S : sphere of radius 1

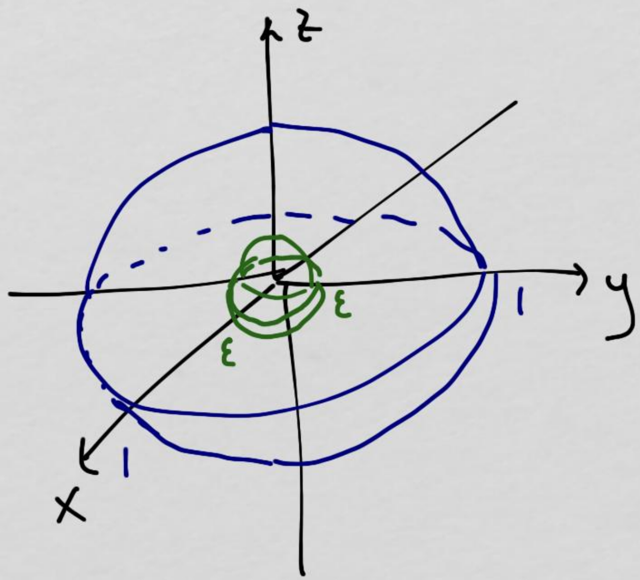
flux ?

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \operatorname{div} \vec{F} \, dV$$

problem : $\vec{F}$ does not exist at $(0,0,0)$, so the surface integral is not affected because S is the surface of the sphere and does not contain $(0,0,0)$, but the volume integral cannot be directly computed. ($(0,0,0)$ is inside the volume))

How can we apply the Divergence Theorem?

imagine that we have a very tiny sphere around the origin, use Divergence Theorem to calculate flux through the region between spheres, then we shrink the inner sphere by taking a limit (radius → 0)



Small sphere: radius ε (epsilon → usually used to denote something small)

the region between the spheres is easy

$$\varepsilon \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi$$

$$\vec{F} = \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}} \qquad \operatorname{div}\vec{F} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2\rho^2}{\rho^3} = \frac{2}{\rho}$$

So, the flux is: $$\int_{\underset{\theta}{0}}^{2\pi}\int_{\underset{\phi}{0}}^{\pi}\int_{\underset{\rho}{\varepsilon}}^{1} \underbrace{\frac{2}{\rho}}_{\mathrm{div}\,\vec{F}}\ \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$= 4\pi(1-\varepsilon^2)$$

now, we take the limit $\varepsilon \to 0$ and we get $\boxed{4\pi}$