15.1 Functions of Several Variables

we are familiar with functions of one variable

for example, $y = f(x) = \underbrace{\sqrt{9-x^2}}_{\text{output}}$

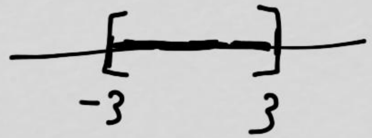
output input

the set of all allowable input values is the <u>domain</u>

for this function, the domain is $9 - x^2 \geq 0$

$-3 \leq x \leq 3$ or $[-3, 3]$

or



for one-variable function, the domain is a line or portions of lines the set of all possible output values is the <u>range</u>

for $f(x) = \sqrt{9-x^2}$ the smallest possible output is 0 (when $x = \pm 3$)

the largest possible output is 3 (when $x = 0$)

so, the range is $0 \le y \le 3$ or $[0, 3]$

almost all of this carry over to functions of two or more variables

(the domain is the one thing that is most different)

for example, $$z = f\underbrace{(x,y)}_{\text{input}} = \underbrace{\sqrt{9-x^2} - \sqrt{25-y^2}}_{\text{output}}$$

output

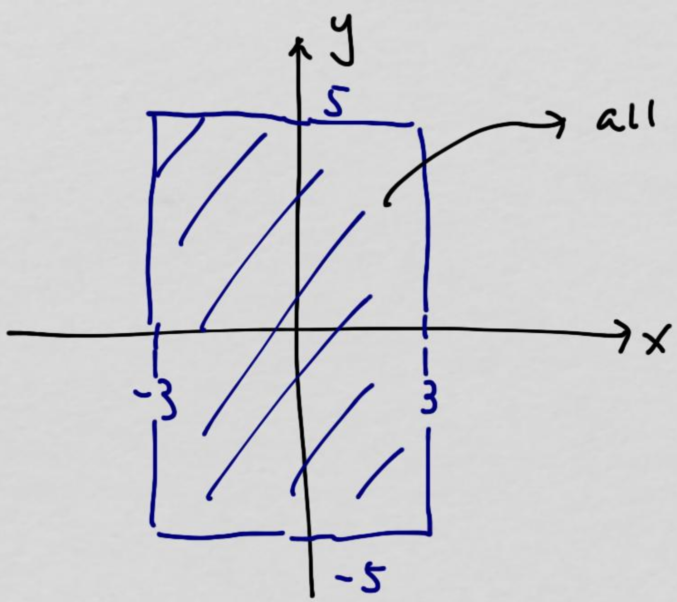
now the inputs are <u>ordered pairs</u> (x, y)

the domain is still the set of all allowable inputs for $z = f(x,y) = \sqrt{9-x^2} - \sqrt{25-y^2}$

we need (x,y) to be such that $9-x^2 \ge 0$ AND $25-y^2 \ge 0$

or $-3 \le x \le 3$ AND $-5 \le y \le 5$

we can visualize this domain as a region in $\mathbb{R}^2$



all (x,y) that can go into $f(x,y) = \sqrt{9-x^2} - \sqrt{25-y^2}$

another way to express this is

$$\boxed{\{(x,y) : -3 \le x \le 3,\ -5 \le y \le 5\}}$$

the output $z = f(x,y) = \sqrt{9-x^2} - \sqrt{25-y^2}$ is still just a single value

so, the collection of possible outputs is not different from a one-variable function

$$f(x,y) = \underbrace{\sqrt{9-x^2}}_{\substack{\text{no smaller}\\ \text{than } 0\\ \text{and no bigger}\\ \text{than } 3}} - \underbrace{\sqrt{25-y^2}}_{\substack{\text{no smaller}\\ \text{than } 0\\ \text{and no bigger}\\ \text{than } 5}}$$

so, the largest possible output is 3 (when $\sqrt{9-x^2} = 3$ and $\sqrt{25-y^2} = 0$)

and the smallest possible output is -5 (when $\sqrt{9-x^2} = 0$ and $\sqrt{25-y^2} = 5$)

range: $-5 \le z \le 3$

or $[-5, 3]$

or $\{ z : -5 \le z \le 3 \}$

we know $z = f(x,y)$ is a surface (cone, paraboloid, etc)

if we take a slice of the surface at $z = z_0$ = constant (or the intersection of $f(x,y)$ with the plane $z = z_0$), we get a level curve or a contour of $f(x,y) = z_0$

(another way to say it: we get a curve that contains all possible (x,y) such that $f(x,y) = z_0$)

this is very similar to the concept of trace

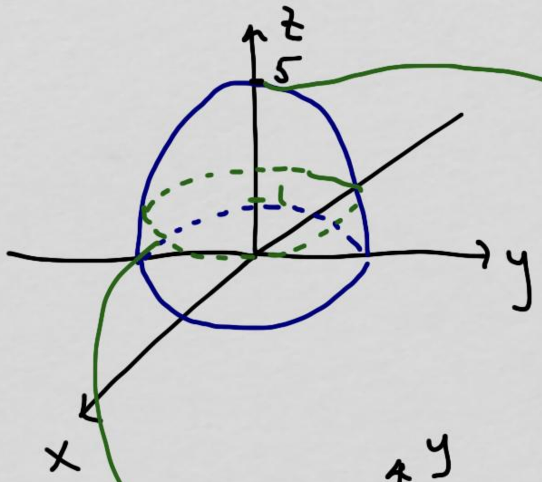
example

$z = f(x,y) = 5 - x^2 - y^2$ (paraboloid)

domain: $-\infty < x < \infty$, $-\infty < y < \infty$

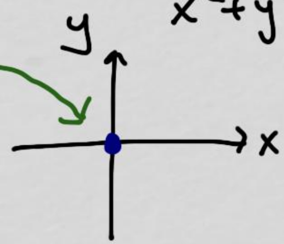
or $\{(x,y) : \mathbb{R}^2\}$

range: $\{z : z \le 5\}$

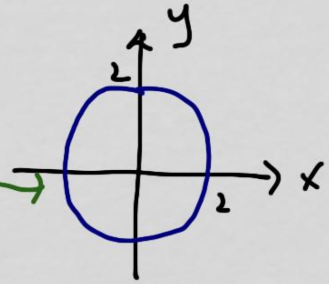


slice at $z=5$ (level curve of $f(x,y)=5$)

$z = f(x,y) = 5 - x^2 - y^2 = 5$



$x^2+y^2=0$ circle of radius 0



slice at $z=1$ (level curve of $z=1$)

$z = f(x,y) = 5 - x^2 - y^2 = 1$

$x^2+y^2=4$ circle of radius 2

if we keep slicing through different z values, we get more cross sections

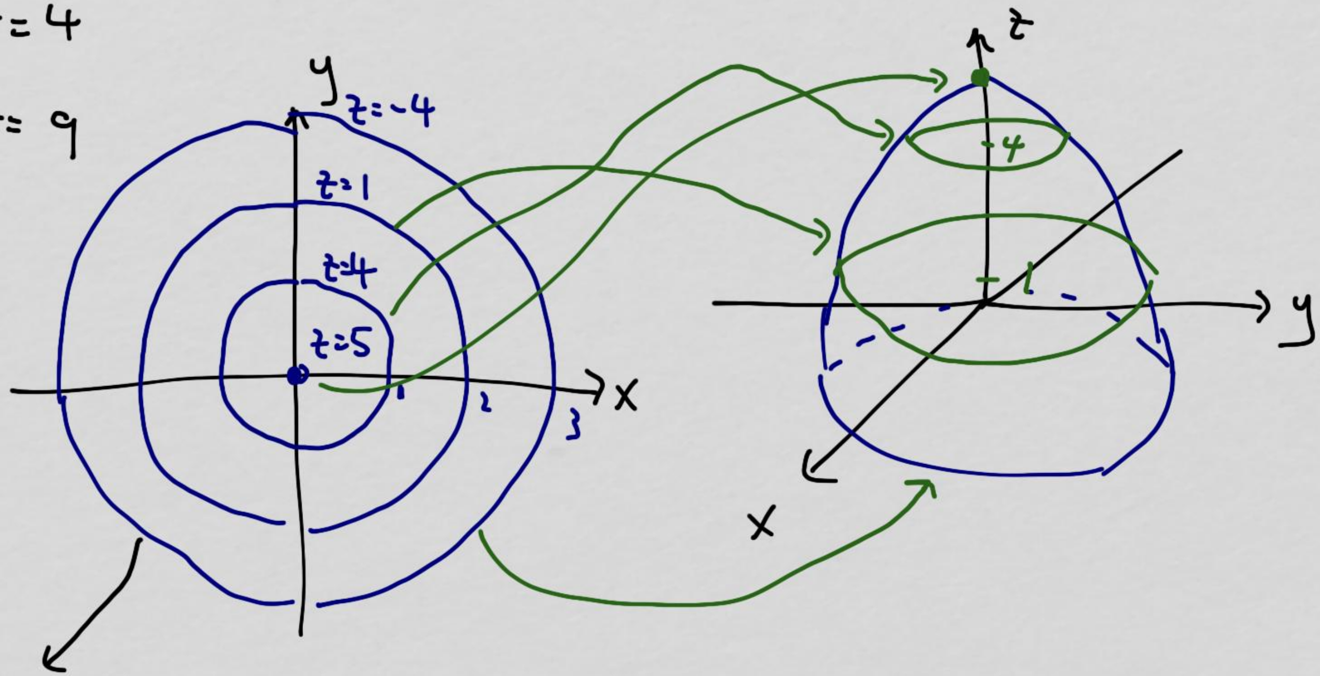
$z = f(x,y) = 5 - x^2 - y^2$

$z = 5 \rightarrow x^2 + y^2 = 0$

$z = 4 \rightarrow x^2 + y^2 = 1$

$z = 1 \rightarrow x^2 + y^2 = 4$

$z = -4 \rightarrow x^2 + y^2 = 9$



collection of all (x,y)
such that $z = -4$

example

$f(x,y) = \sin(xy)$

domain : $\{ (x,y) : \mathbb{R}^2 \}$

range : $\{ z : -1 \le z \le 1 \}$

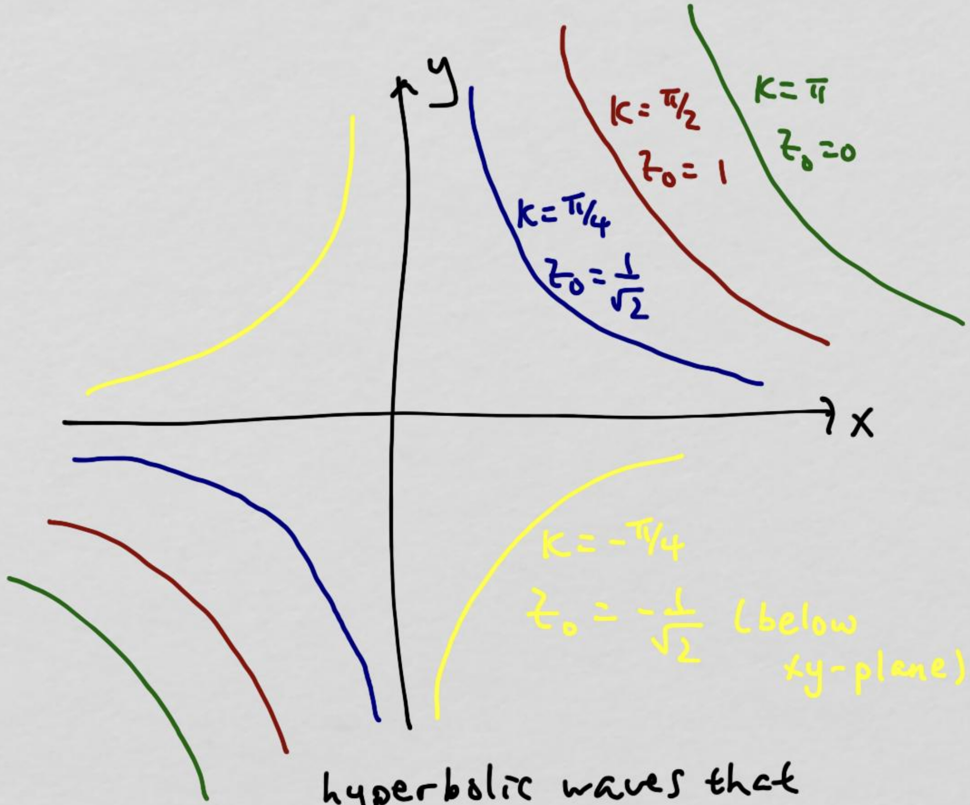
the level curves of this are a bit more tricky than the last one

if $z = f(x,y) = z_0$

then $z_0 = \underbrace{\sin(xy) = \text{constant}}_{\text{means } xy = \text{constant} = k}$

$xy = k$ is easy to graph

$$y = \frac{k}{x}$$



$$y=\frac{k}{x} \rightarrow z_0=\sin(k)$$

height through which we slice

if $k=\frac{\pi}{4}$ $\left(z_0=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\right)$

$$y=\frac{\pi/4}{x}$$

if $k=\frac{\pi}{2}$ $\left(z_0=\sin\left(\frac{\pi}{2}\right)=1\right)$

$$y=\frac{\pi/2}{x}$$

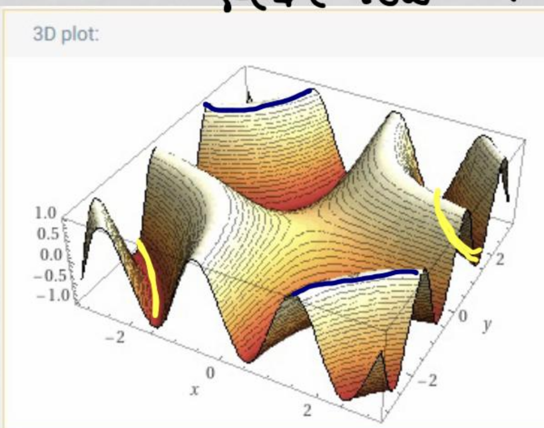
if $k=\pi$ $\left(z_0=\sin(\pi)=0\right)$

$$y=\frac{\pi}{x}$$

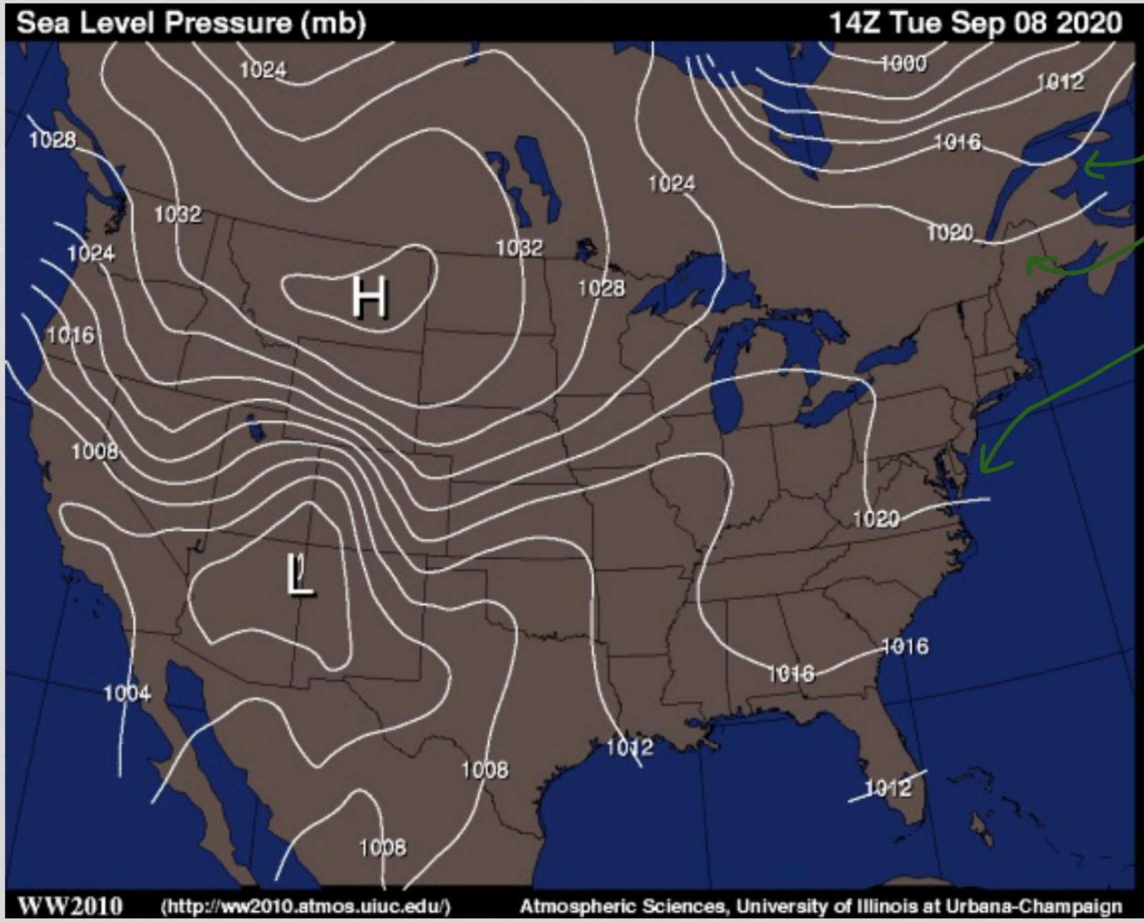
if $k=-\frac{\pi}{4}$ $\left(z_0=\sin\left(-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}\right)$

$$y=\frac{-\pi/4}{x}$$

hyperbolic waves that
start high in QI and QIII
start low in QII and QIV



we see contours / level curves in real-life applications such as weather forecasting



lines of constant pressure

(isobars)

the same

pressure