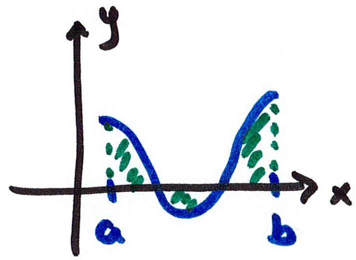
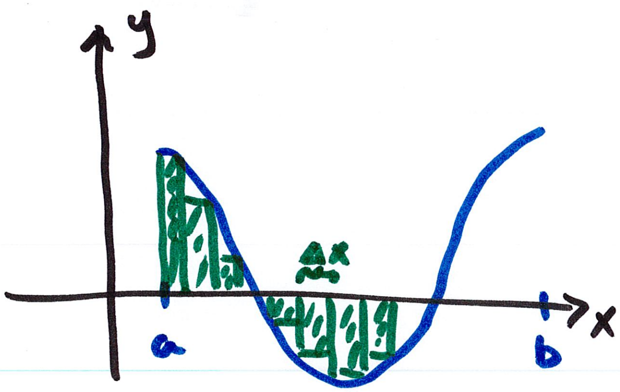
16.1 Double Integrals over Rectangular Regions

if $y=f(x)$ $a \le x \le b$



then $\int_a^b f(x)dx$ gives the net area bounded by $f(x)$ and x-axis

~~recall~~ remember $\int_a^b f(x)dx$ is really the sum of areas of infinitely-many rectangles

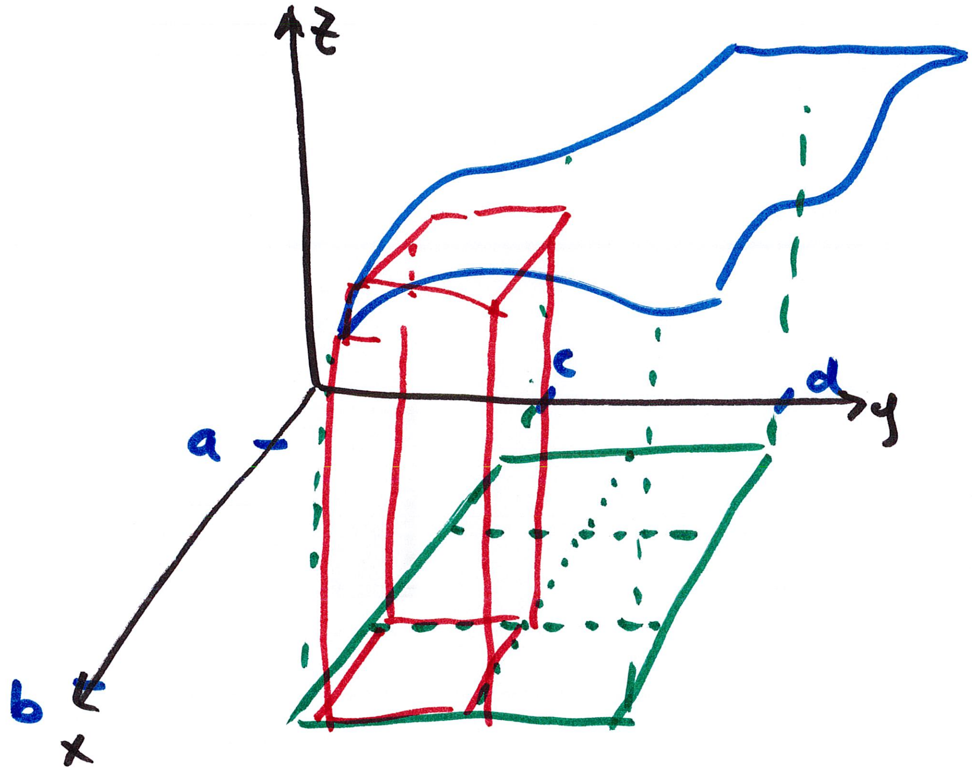


each rectangle has width Δx
and height $f(x_i)$

sample point
(left end, mid pt, etc)

then $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \underbrace{f(x_i)\Delta x}_{\text{area of each rectangle}} = \int_a^b f(x)dx$

$z = f(x,y)$ is a surface. We can use the same basic idea to find the volume under $f(x,y)$ above $a \le x \le b$ $c \le y \le d$ $([a,b] \times [c,d])$

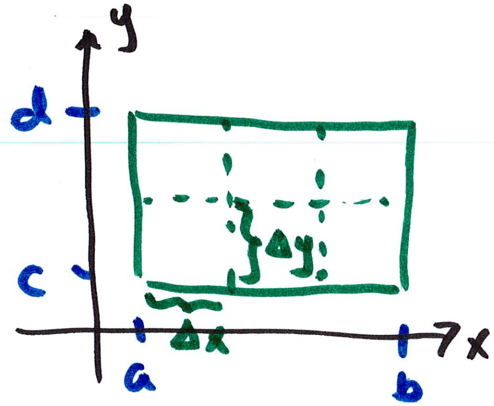


to find the volume under this surface above the green rectangle chop into rectangular boxes

each red box has volume equal to

$f(x_i, y_j)\,\Delta x\,\Delta y = \Delta A$

to find total volume under surface over $[a,b] \times [c,d]$ we sum up the boxes' volumes



and then shrink $\Delta x \to dx$ $\Delta y \to dy$ and $N \to \infty$

$\nwarrow$ # of boxes

total volume: $\displaystyle \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$

Subdivisions in x Subdivisions in y

$$= \boxed{\int_a^b \int_c^d f(x,y)\,dA}$$

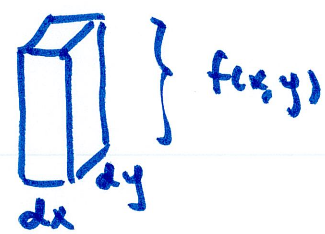
double integral

$dA = dx\,dy$ or $dy\,dx$

$a \le x \le b$, $c \le y \le d$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

however, the volume of each box



$f(x,y)\,dx\,dy$

OR

$f(x,y)\,dy\,dx$

is also

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

but, switching the order of integration is usually NOT this easy if the region is NOT a rectangle $[a,b] \times [c,d]$

how to evaluate double integrals?

example

$$\int_0^1 \int_0^2 (3-x-y)\,dy\,dx$$

basic process: integrate <u>inside-out</u>

↳ inside integral, dy means y is variable, x treated as constant

$$= \int_0^1 \left(3y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=2}\right) dx$$

$$= \int_0^1 (6-2x-2-0)\,dx = \int_0^1 (4-2x)\,dx$$

$$= 4x - x^2 \Big|_0^1 = \boxed{3}$$

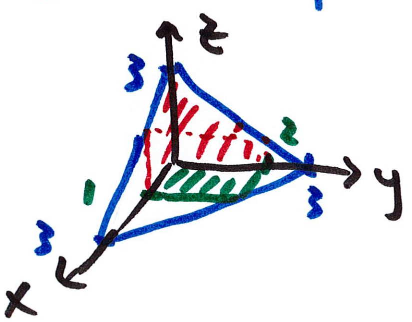
Since both x and y are bounded by constants, we are integrating over a rectangle, so the order of integration does not matter

$$\int_0^1 \int_0^2 (3-x-y)\,dy\,dx = \int_0^2 \underbrace{\int_0^1 (3-x-y)\,dx}_{dx:\ y \text{ is constant}}\,dy$$

$$= \int_0^2 3x - \frac{1}{2}x^2 - yx \Big|_{x=0}^{x=1} dy$$

$$= \int_0^2 \left(3 - \frac{1}{2} - y\right) dy = \frac{5}{2}y - \frac{1}{2}y^2 \Big|_0^2 = 5 - 2 = \boxed{3}$$

physical interpretation: volume under $f(x,y) = 3-x-y$ above the rectangle $0 \le x \le 1$ $\quad 0 \le y \; 2$



Switching order can sometimes make integration easier (or harder)

<u>example</u> $\displaystyle \int_0^1 \underbrace{\int_0^2 y^5 x^2 e^{x^3y^3}\,dx}_{dx:\ y \text{ is constant}}\,dy$

$$\int_0^2 y^5x^2e^{x^3y^3}\,dx = y^5\int_0^2 \underline{\underline{x^2}}\,e^{x^3y^3}\,\underline{\underline{dx}}$$

$u = x^3y^3$

$du = 3x^2y^3\,dx$

$x^2dx = \frac{1}{3y^3}\,du$

$$= y^5\int_{x=0}^{x=2} e^u\cdot\frac{1}{3y^3}\,du = y^5\cdot\frac{1}{3y^3}\,e^u\Big|_{x=0}^{x=2} = \frac{1}{3}y^2e^{x^3y^3}\Big|_{x=0}^{x=2}$$

$$= \frac{1}{3}y^2e^{8y^3} - \frac{1}{3}y^2$$ this is the inner integral

finish the outer integral

$$\int_0^1 \left(\frac{1}{3}y^2e^{8y^3} - \frac{1}{3}y^2\right)dy$$

$$= \underbrace{\int_0^1 \frac{1}{3}y^2e^{8y^3}\,dy}_{\substack{u=8y^3 \\ du=24y^2dy \\ \vdots \\ \text{etc}}} - \underbrace{\int_0^1 \frac{1}{3}y^2\,dy}_{\text{easy}} = \cdots = \boxed{\frac{1}{72}e^8 - \frac{1}{8}}$$

what happens if we switch order?

$$\int_0^2 \underbrace{\int_0^1 y^5x^2e^{x^3y^3}\,dy}_{dy:\ x\ \text{const.}}\,dx$$

$$x^2\int_0^1 y^5e^{x^3y^3}\,dy$$ by parts (harder than original order)

same idea for $z = f(x,y)$

volume under $z = f_{avg}$ over $[a,b] \times [c,d]$

to be the same as $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

$$= \iint_R f(x,y)\,dA$$

$R: \{(x,y) : a \le x \le b,\ c \le y \le d\}$

area of rectangle $[a,b] \times [c,d]$

$$f_{avg} \cdot A = \iint_R f(x,y)\,dA$$

$$\boxed{f_{avg} = \frac{1}{A} \iint_R f(x,y)\,dA}$$

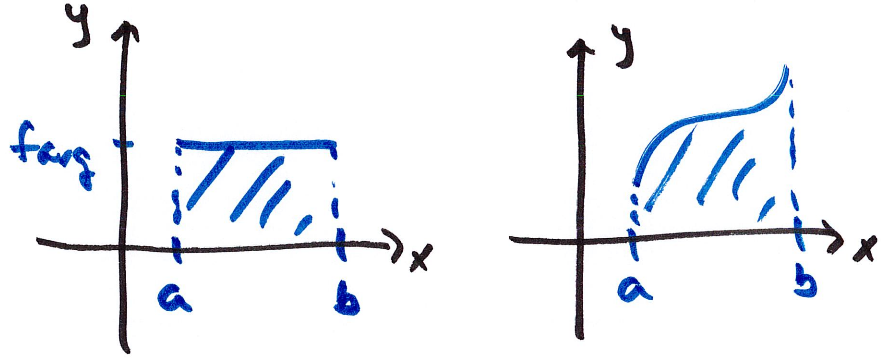
area of rectangle we integrate over

recall if $y = f(x)$, $a \le x \le b$

then the average value of $f(x)$ over $a \le x \le b$ is

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

this comes from $f_{avg} \cdot (b-a) = \int_a^b f(x)\,dx$



same areas.

additional example:

$$\int_0^1 \underbrace{\int_0^{\pi/3} x^2 \cos(xy)\,dx}_{dx:\ y\text{ const.}}\,dy$$

$\int_0^{\pi/3} x^2 \cos(xy)\,dx$ requires two rounds of integration by parts

$u = x^2 \quad dv = \cos(xy)\,dx$

$\vdots$

etc

the other order: $\displaystyle\int_0^{\pi/3} \underbrace{\int_0^1 x^2 \cos(xy)\,dy}_{dy:\ x\text{ const.}}\,dx$

$$\int_0^1 x^2 \cos(xy)\,dy = x^2 \cdot \sin(xy) \cdot \frac{1}{x} \Big|_{y=0}^{y=1} = x \sin x$$

then $\int_0^{\pi/3} x \sin x\,dx$ only needs one round of integration by parts