16.2 Double Integrals over General Regions

last time: rectangular regions $R = \{(x,y) : a \le x \le b,\ \ c \le y \le d\}$

we could swap integration order at will

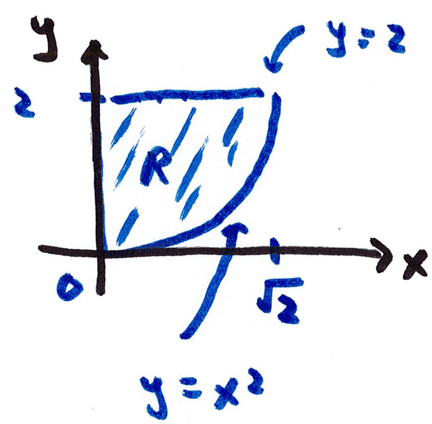
$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

However, swapping order arbitrary is only possible with rectangular regions

<u>example</u> $f(x,y) = xy^2$ $\quad R = \{(x,y) : 0 \le x \le \sqrt{2},\ \ x^2 \le y \le 2\}$

bottom of R   top of R

sketch R



Basic Rule: integrate the variable with constants <u>LAST</u>

(the <u>outside</u> integral)

here, we integrate y first (inside)

$$\int_0^{\sqrt{2}} \int_{x^2}^{2} x y^2 \, dy \, dx$$

$$x \qquad y$$

$$= \int_0^{\sqrt{2}} \frac{1}{3} x y^3 \Big|_{y=x^2}^{y=2} dx = \int_0^{\sqrt{2}} \left(\frac{8}{3} x - \frac{1}{3} x^7 \right) dx = \frac{8}{6} x^2 - \frac{1}{24} x^8 \Big|_0^{\sqrt{2}}$$

$$= \boxed{2}$$

what if we used the wrong order?

$$\int_{x^2}^{2} \int_0^{\sqrt{2}} x y^2 \, dx \, dy = \int_{x^2}^{2} \frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{2}} dy$$

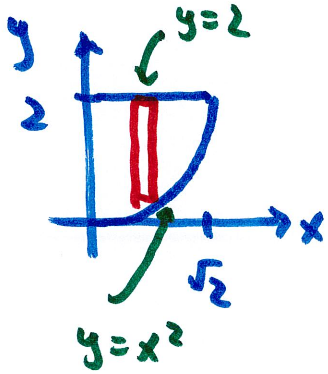
$$y \qquad x$$

what to do with this?

$$= \int_{x^2}^{2} y^2 \, dy = \frac{1}{3} y^3 \Big|_{x^2}^{2} = \frac{8}{3} - \frac{1}{3} x^6$$

we <u>can</u> still swap the order, but we need to reformulate the region

$R = \{(x,y) : 0 \le x \le \sqrt{2},\ x^2 \le y \le 2\}$

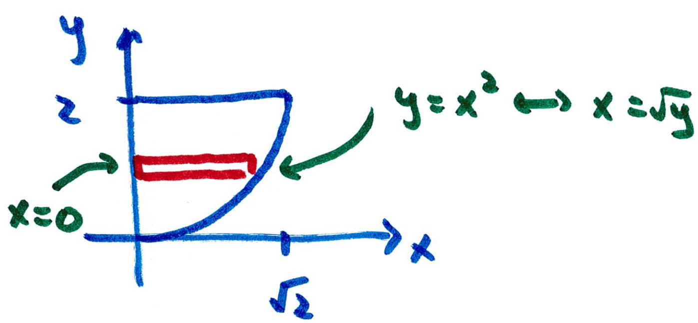


this is a <u>Type I</u> region: x bounded by constants

y bounds form the top and bottom of a typical element

(height of a small rectangle)

to swap order, need to change to a <u>Type II</u> region: y bounded by constants



determine curves touching ends of element

$$\boxed{R = \{(x,y) : \underbrace{0}_{\text{left}} \le x \le \underbrace{\sqrt{y}}_{\text{right}},\ 0 \le y \le 2\}}$$

new integral: $\displaystyle\int_0^2 \int_0^{\sqrt{y}} xy^2 \, dx \, dy = \int_0^2 \frac{1}{2}x^2y^2 \Big|_{x=0}^{x=\sqrt{y}} dy = \int_0^2 \frac{1}{2} y^3 \, dy$

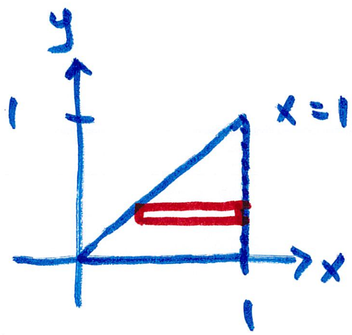
y x

$$= \frac{1}{8} y^4 \Big|_0^2 = \frac{16}{8} = \boxed{2}$$

example

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

as provided: $R = \{(x,y):\ \underbrace{y}_{\text{left: } x = y} \le x \le 1,\ \ 0 \le y \le 1\}$ Type II

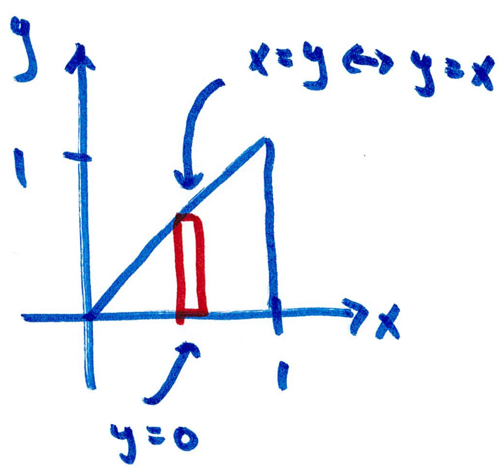


swap? check out the first integral first

$$\int_y^1 e^{x^2}\,dx$$

this is not something we can integrate!

Since we can't even get past the first integral, swap order



$0 \le y \le x$, $\quad 0 \le x \le 1$

$R = \{(x,y):\ 0 \le x \le 1,\ \ 0 \le y \le x\}$

"new" integral:

$$\int_{\underset{x}{0}}^1 \int_{\underset{y}{0}}^x e^{x^2}\,dy\,dx = \int_0^1 y e^{x^2}\Big|_{y=0}^{y=x}\,dx$$

$$= \int_0^1 x e^{x^2}\, dx = \cdots = \boxed{\tfrac{1}{2}(e-1)}$$

subs $u = x^2$
$du = 2x\,dx$

<u>example</u>

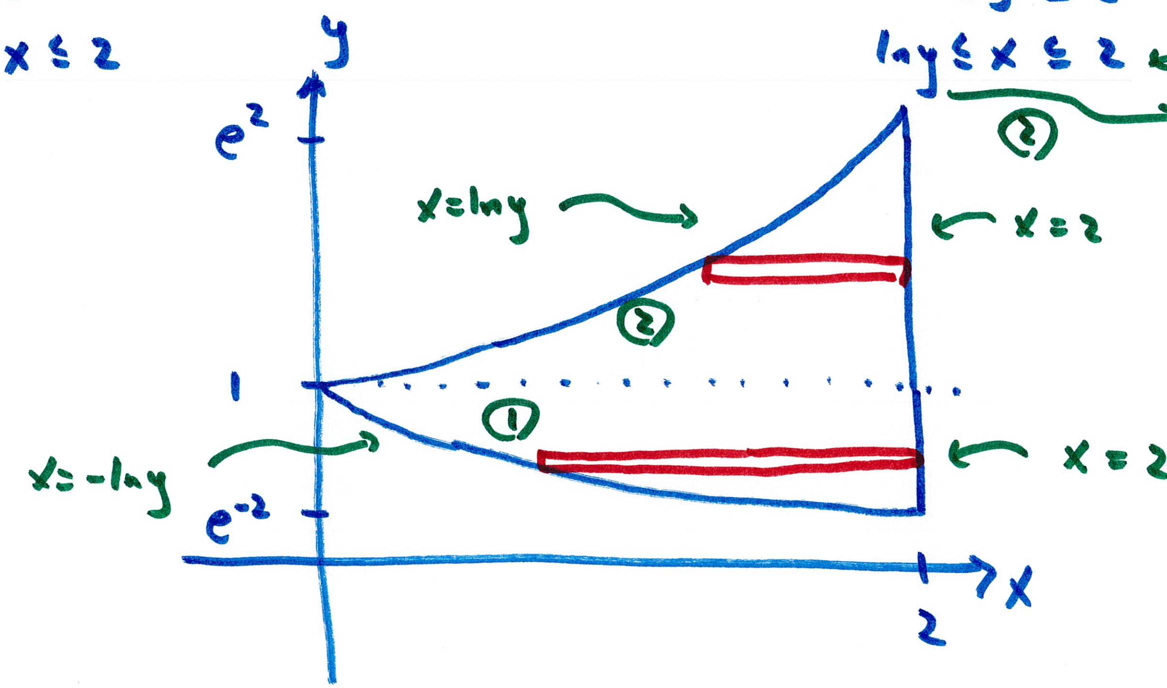
$$\underbrace{\int_{e^{-2}}^{1} \int_{-\ln y}^{2} f(x,y)\, dx\, dy}_{} + \underbrace{\int_{1}^{e^2} \int_{\ln y}^{2} f(x,y)\, dx\, dy}_{}$$

turn this into integral(s) of the other type

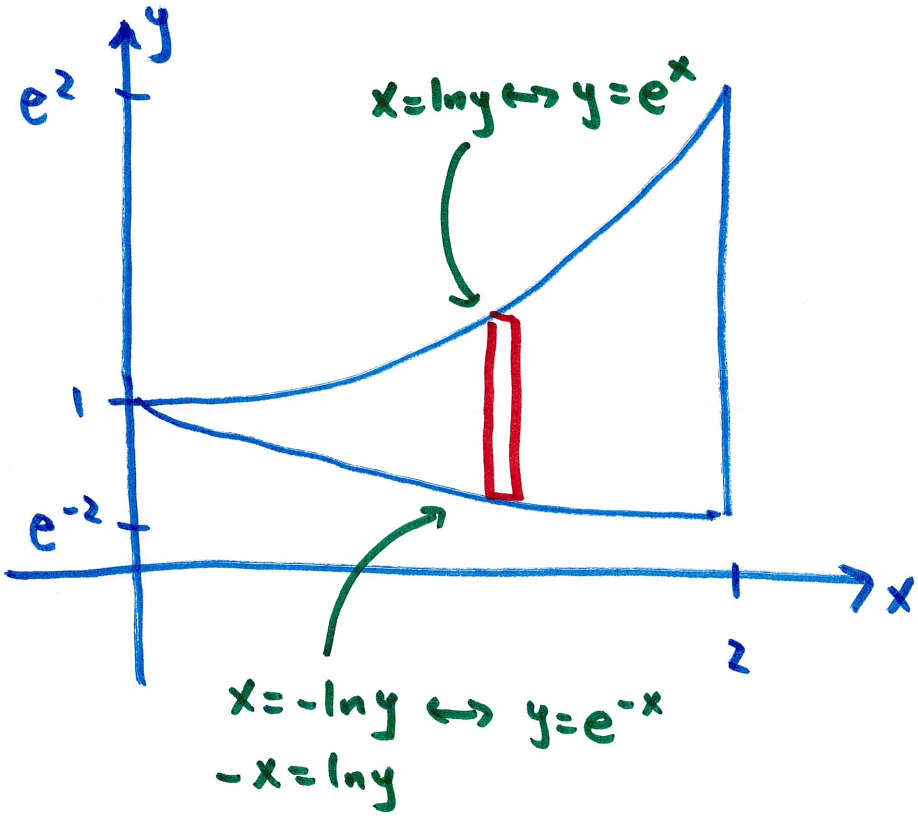
$e^{-2} \le y \le 1$ | $1 \le y \le e^2$
$-\ln y \le x \le 2$ | $\ln y \le x \le 2$ ← right : $x = 2$

(1) | (2) → left : $x = \ln y$

left curve
$x = -\ln y$



change to Type I

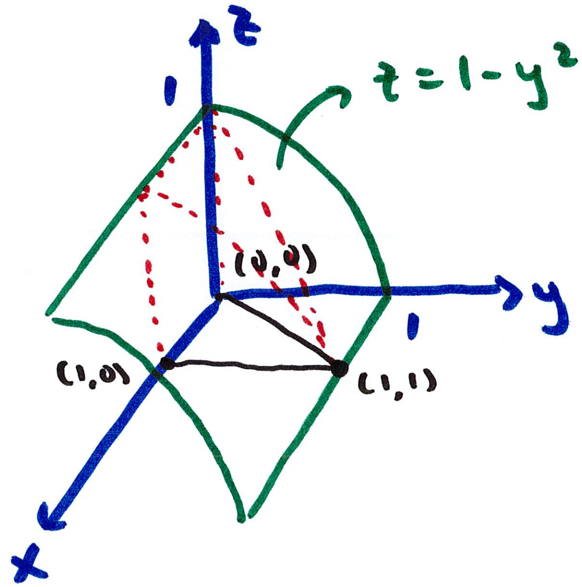


notice that the top/bottom curves never change so only one integral is needed

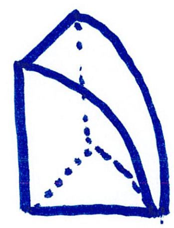
so, as Type I: $R = \{ (x,y) : 0 \le x \le 2,\ e^{-x} \le y \le e^{x} \}$

equivalent integral : $\displaystyle\int_{x=0}^{2} \int_{y=e^{-x}}^{e^{x}} f(x,y)\,dy\,dx$

example Find the volume of the solid bounded above by $z = 1-y^2$ and above the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ on the xy-plane

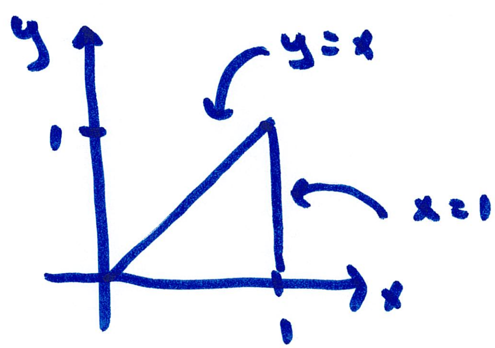


we want volume of thing that's below the part of $z=1-y^2$ with the triangular "shadow"



volume = ?

let's look at the "shadow"



as Type I: $0 \le x \le 1$, $0 \le y \le x$

as Type II: $y \le x \le 1$, $0 \le y \le 1$

to find volume, we integrate the height over the "shadow"

Type I : $\int_0^1 \int_0^x (1-y^2)\,dy\,dx$

Type II: $\int_0^1 \int_y^1 (1-y^2)\,dx\,dy$

both are easy (answer is $\frac{5}{12}$)