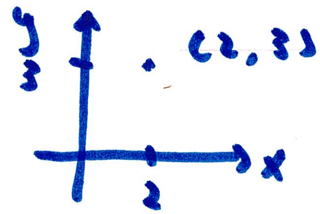
16.3 Double Integrals in Polar Coordinates

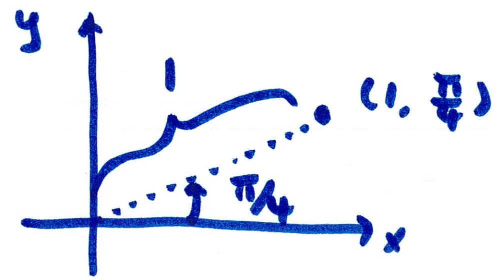
rectangular / Cartesian : (x, y)



polar : (r, θ)

r: displacement from origin

θ: angle of line from origin to the point measured from positive x-axis

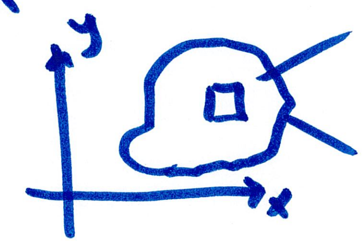


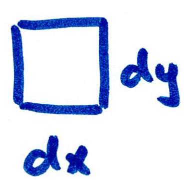
conversion : $x^2 + y^2 = r^2$

$x = r\cos\theta$

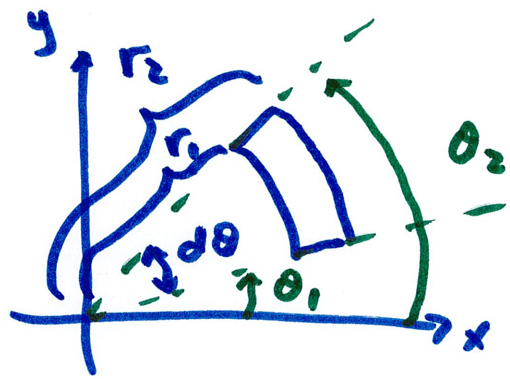
$y = r\sin\theta$

In Cartesian, $\iint_R f(x,y)\,dA$ $\qquad dA = dx\,dy = dy\,dx$

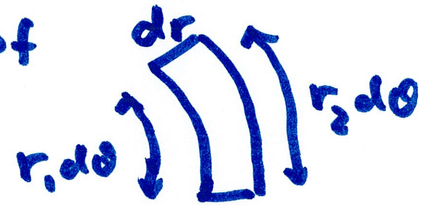




In polar $\iint_R f(r,\theta)\,dA$



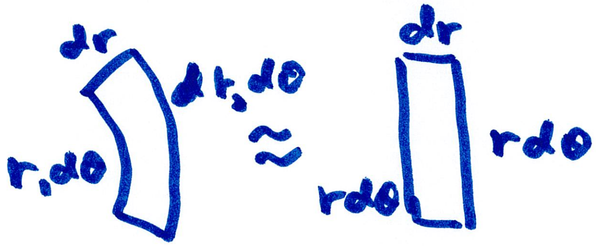
dA is area of



dr : difference between r_1, r_2

$d\theta$: difference between θ_1, θ_2

notice if dr and $d\theta$ are very small, then $r_1 \approx r_2 = r$

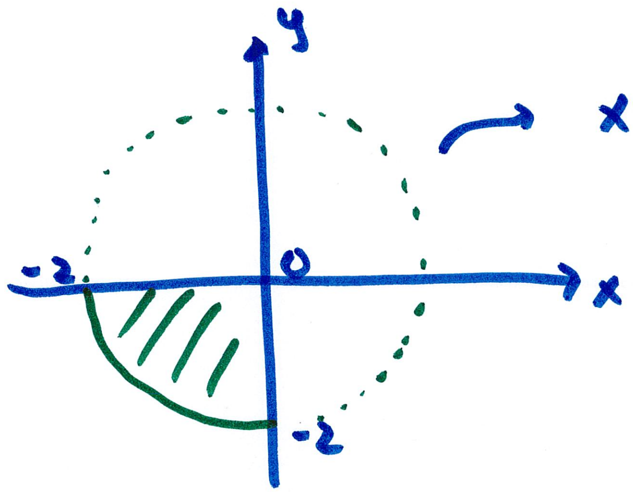


so, in polar, $\boxed{dA = r\,dr\,d\theta}$

example

$$\iint_R \cos(x^2+y^2)\,dA$$

R: circle centered at origin with radius 2, in 3rd quadrant



$x^2+y^2=2^2$

$y = \pm\sqrt{4-x^2}$

in QIII, $y = -\sqrt{4-x^2}$

in Cartesian, expressing R as a Type I region,

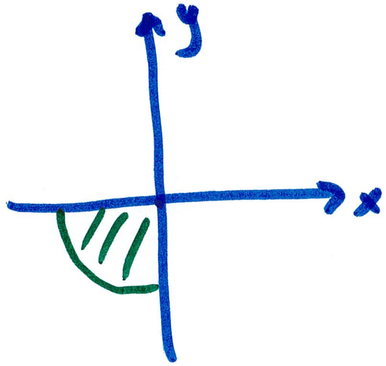
$$-2 \le x \le 0$$

$$-\sqrt{4-x^2} \le y \le 0$$

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0} \cos(x^2+y^2)\,dy\,dx$$

difficult in Cartesian

polar is effective w/ circles or circle-like things express same integral in polar



$$R = \{(r,\theta) : \underbrace{0 \le r \le 2, \quad \pi \le \theta \le \tfrac{3\pi}{2}}_{\text{both bounded by constants any order is fine}}\}$$

integral is

$$\int_{\pi}^{\frac{3\pi}{2}} \int_{0}^{2} \cos(r^2)\, \overbrace{r\,dr\,d\theta}^{dA}$$

θ $\quad$ r $\quad$ $x^2+y^2=r^2$

$$= \int_{\pi}^{\frac{3\pi}{2}} \underbrace{\int_{0}^{2} r\cos(r^2)\,dr}_{\substack{u=r^2 \\ du=2r\,dr}}\,d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}\sin(u)\Big|_{u=0}^{u=4} d\theta = \dots = \boxed{\frac{\pi}{4}\sin(4)}$$

example

$$\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

terrible in Cartesian

but bounds for y look like circles

change to polar

$$0 \le x \le 1$$

$$\underbrace{\sqrt{x-x^2}} \le y \le \underbrace{\sqrt{1-x^2}}$$

$y = \sqrt{x-x^2}$

$y^2 = x - x^2$

$x^2 - x + y^2 = 0$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

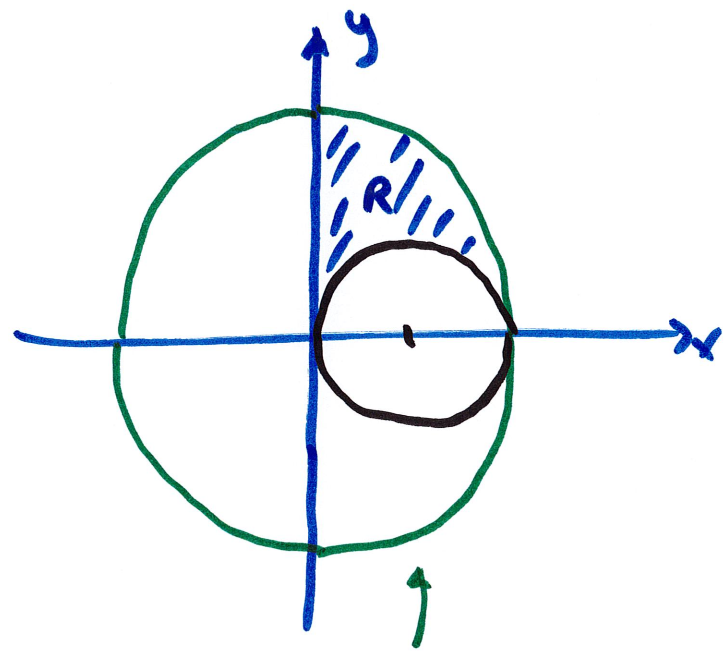
circle radius $\frac{1}{2}$

centered at $(\frac{1}{2}, 0)$

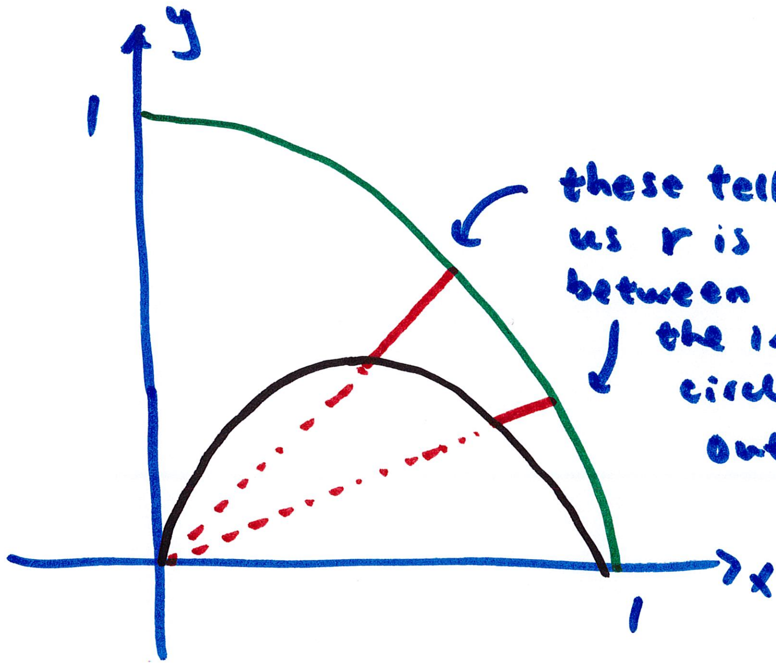
upper half of

circle radius 1

centered at $(0,0)$



circle radius 1



these tell us r is between the inner circle and outer circle

express R in polar

$$\boxed{0 \leq \theta \leq \pi/2}$$

inner circle $\leq r \leq$ outer circle

express circles in polar

inner : $y^2 = x - x^2$

$$x^2 + y^2 = x$$

$$r^2 = r\cos\theta \rightarrow r = \cos\theta$$

outer : $r = 1$

so, $\boxed{\cos\theta \leq r \leq 1}$

$R = \{(r,\theta) : \cos\theta \leq r \leq 1, \quad 0 \leq \theta \leq \pi/2\}$

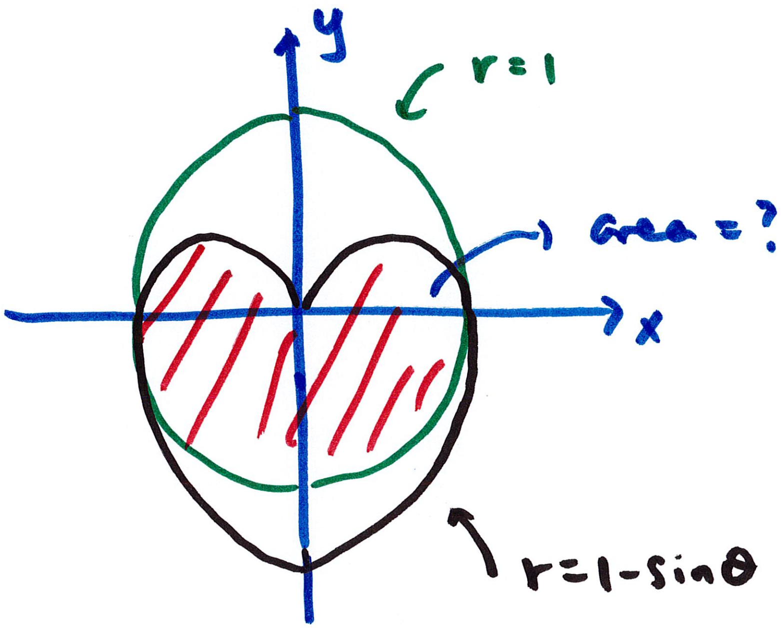
integrate r first (not bounded by constants)

$$\underset{\theta}{\int_0^{\pi/2}} \underset{r}{\int_{\cos\theta}^{1}} \underset{\overset{\uparrow}{x^2+y^2}}{(r^2)^{3/2}} \, \underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\pi/2} \int_{\cos\theta}^{1} r^4 \, dr \, d\theta$$

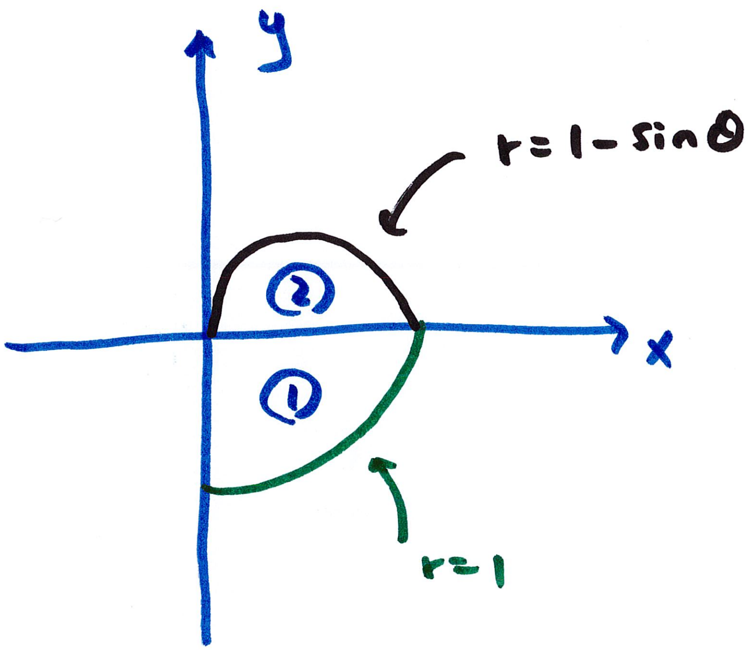
$$= \dots = \boxed{\frac{\pi}{10} - \frac{8}{75}}$$

if problems involve polar equations, do integral in polar (usually)

(for example, "find area between $r = 1 - \sin\theta$ and $r = 1$"}



take advantage of symmetry, find area of right half, then multiply by 2



1 : $-\frac{\pi}{2} \le \theta \le 0$

$0 \le r \le 1$

2 : $0 \le \theta \le \frac{\pi}{2}$

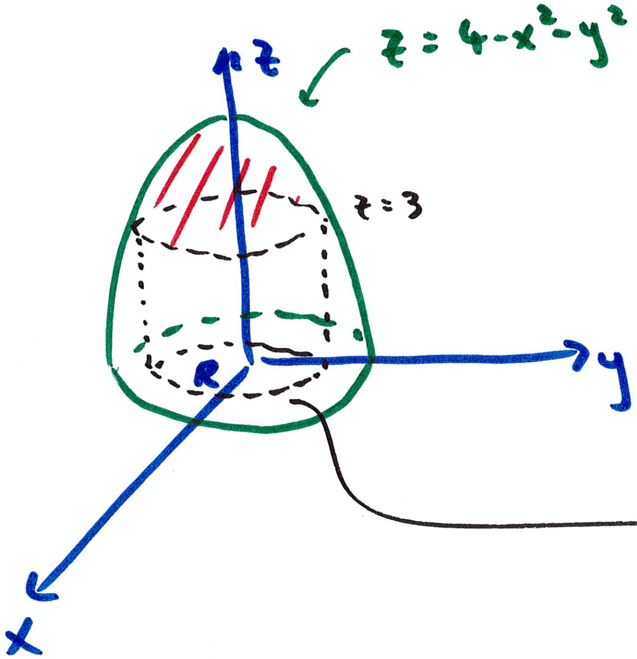
$0 \le r \le 1 - \sin\theta$

integrate dA to find area

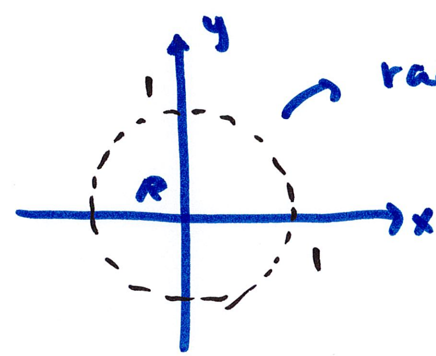
$$\underbrace{\int_{-\frac{\pi}{2}}^{0} \int_{0}^{1} r\,dr\,d\theta}_{1} + \underbrace{\int_{0}^{\pi/2} \int_{0}^{1-\sin\theta} r\,dr\,d\theta}_{2} = \dots = \frac{5\pi - 8}{8}$$

don't forget to multiply by 2 to find area of whole heart.

example Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ ~~and~~ above $z = 3$



we need to integrate the height from the "floor" ($z = 3$) to the "ceiling" ($z = 4 - x^2 - y^2$) above the region that is the "shadow" of the part we want



radius is $z = 4 - x^2 - y^2$ at $z = 3$

$3 = 4 - x^2 - y^2$

$x^2 + y^2 = 1$

so radius 1

so, $R = \{(r, \theta) : 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}$

integrate $(4 - x^2 - y^2) - 3 = (4 - r^2) - 3 = 1 - r^2$ over R

$$\int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \ldots = \boxed{\frac{\pi}{2}}$$