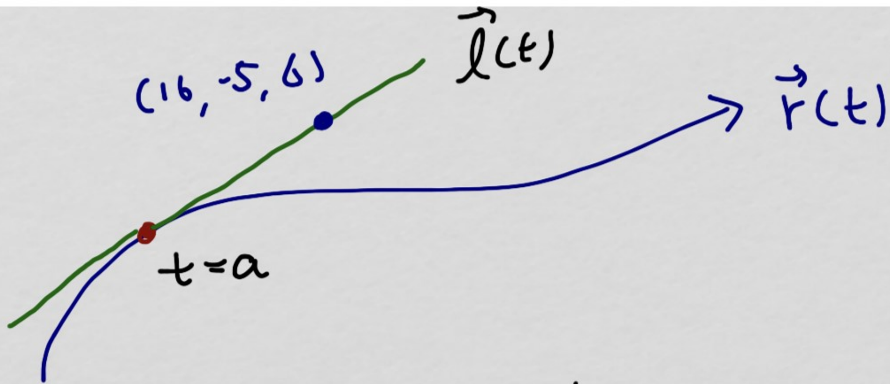
At what point on the curve $\mathbf{r}(t) = < t^2, -t, 6 >$, $t \geq 0$ does the line tangent to that point also go through the point $(16, -5, 6)$?

a) $< 0, 0, 6 >$ b) $< 1, -1, 6 >$ c) $< 4, -2, 6 >$ d) $< 16, -4, 6 >$ e) $< 36, -6, 6 >$



find a such that

$\vec{l}(t)$ goes through $(16, -5, 6)$

$$\vec{l}(t) = \underbrace{\vec{r}(a)}_{\text{position vector to point at } t=a} + \underbrace{\vec{r}\,'(a)}_{\text{direction vector}}\, t$$

direction vector $\vec{v} = \vec{r}\,'(a)$

$\vec{r}\,'(t) = \langle 2t, -1, 0 \rangle$ at $t = a$, $\vec{r}\,'(a) = \underbrace{\langle 2a, -1, 0 \rangle}$

$\vec{r}(a) = \langle a^2, -a, 6 \rangle$

$$\vec{l}(t) = \langle a^2, -a, 6\rangle + t\langle 2a, -1, 0\rangle$$

$$= \langle a^2 + 2at, -a-t, 6\rangle$$

goes through $(16, -5, 6)$ at some t

$$\vec{l}(t) = \langle a^2 + 2at, -a-t, 6\rangle = \langle 16, -5, 6\rangle$$

$$a^2 + 2at = 16$$

$$-a - t = -5$$

$\hookrightarrow$ $t = 5 - a$ (t when $\vec{l}$ goes through $(16, -5, 6)$)

sub into $a^2 + 2at = 16$

$$a^2 + 2a(5-a) = 16$$

$$a^2 + 10a - 2a^2 - 16 = 0$$

$$-a^2 + 10a - 16 = 0$$

$$a^2 - 10a + 16 = 0$$

$$(a-8)(a-2) = 0$$

$a = 2$ or $a = 8$

$a = 2$: point of tangency is $\vec{r}(2) = \langle 4, -2, 6 \rangle$

$a = 8$ " " $\vec{r}(8) = \langle 64, -8, 6 \rangle$

choice C

A smooth parametrization of the semicircle which passes through the points $(1,0,5)$, $(0,1,5)$ and $(-1,0,5)$ is

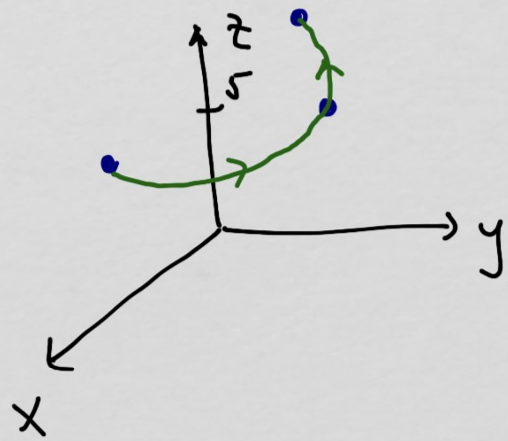
A. $\vec{r}(t) = \sin t\vec{i} + \cos t\vec{j} + 5\vec{k}, 0 \le t \le \pi$

B. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, 0 \le t \le \pi$

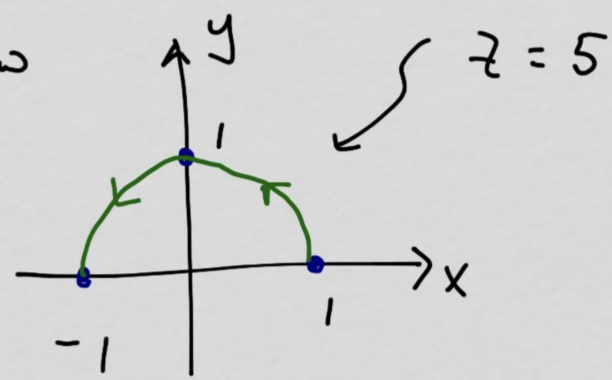
C. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

D. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, 0 \le t \le \frac{\pi}{2}$

E. $\vec{r}(t) = \sin t + \cos t\vec{j} + 5\vec{k}, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$



top view



circle radius 1 : $x = r\cos t$, $y = r\sin t$

$\vec{r}(t) = \langle \cos t, \sin t, 5 \rangle$

$0 \le t \le \pi$

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For the function $f(x, y) = x^2 y$, find a unit vector $\vec{u}$ for which the directional derivative $D_{\vec{u}} f(2, 3)$ is zero.

A. $\vec{i} + 3\vec{j}$ B. $\frac{i+3\vec{j}}{\sqrt{10}}$ C. $\vec{i} - 3\vec{j}$ D. $\frac{i-3\vec{j}}{\sqrt{10}}$ E. $\frac{3\vec{i}-\vec{j}}{\sqrt{10}}$.

$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$

unit vector

$f(x,y) = x^2 y \qquad \vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2xy, x^2 \rangle$

$\vec{\nabla} f(2,3) = \langle 12, 4 \rangle$

$\vec{u} = \langle a, b \rangle$

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = 0 \rightarrow \langle 12, 4 \rangle \cdot \langle a, b \rangle = 0$

$12a + 4b = 0 \rightarrow b = -3a$

$\vec{v} = \langle 1, -3 \rangle \rightarrow \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, -3 \rangle}{|\langle 1, -3 \rangle|} = \frac{\langle 1, -3 \rangle}{10}$

let $a = 1$ $\qquad b = -3a$

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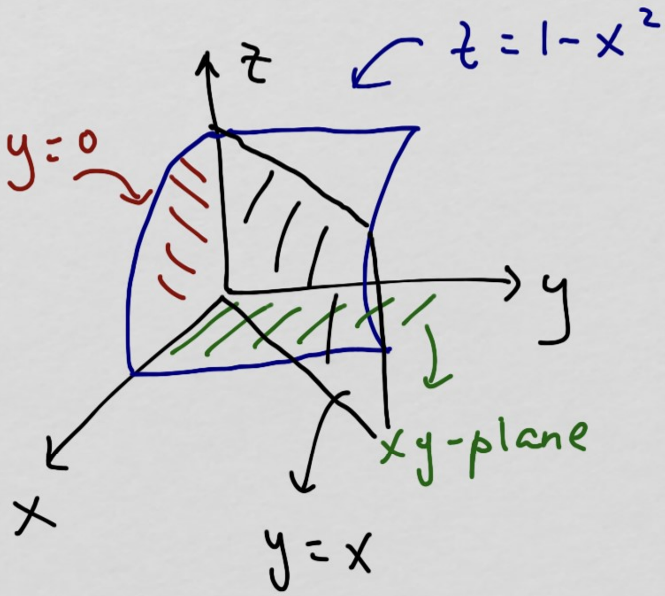
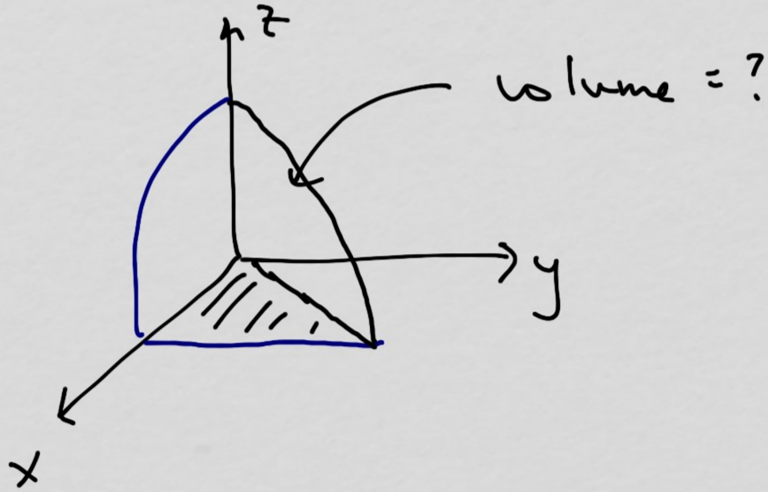
The volume of the solid region in the first octant bounded above by the parabolic sheet $z = 1 - x^2$, below by the xy plane, and on the sides by the planes $y = 0$ and $y = x$ is given by the double integral

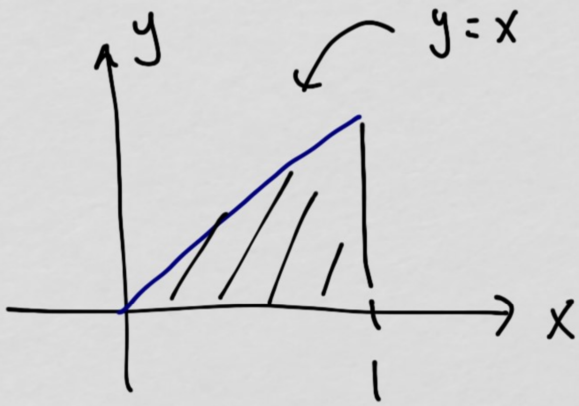
A. $\int_0^1 \int_0^x (1-x^2)dydx$ B. $\int_0^1 \int_0^{1-x^2} x\,dydx$ C. $\int_{-1}^1 \int_{-x}^x (1-x^2)dydx$

D. $\int_0^1 \int_x^0 (1-x^2)dydx$ E. $\int_0^1 \int_x^{1-x^2} dydx.$





all choices have dydx → look at projection onto xy-plane to start



"floor"

dy dx means x last (bounded by constants)

$$0 \le x \le 1$$

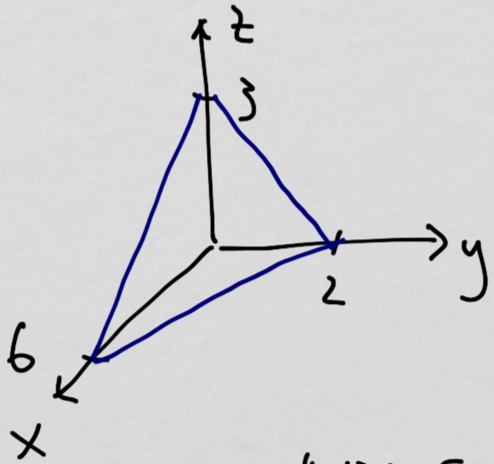
$$0 \le y \le x$$

"ceiling" $z = 1 - x^2$

$$\text{volume} = \int_0^1 \int_0^x (1-x^2)\,dy\,dx$$

Find the area of the portion of the plane $x + 3y + 2z = 6$ that lies in the first octant.

A. $3\sqrt{11}$ B. $6\sqrt{7}$ C. $6\sqrt{14}$ D. $3\sqrt{14}$ E. $6\sqrt{11}$.



area of surface : $S = \iint_D dS$

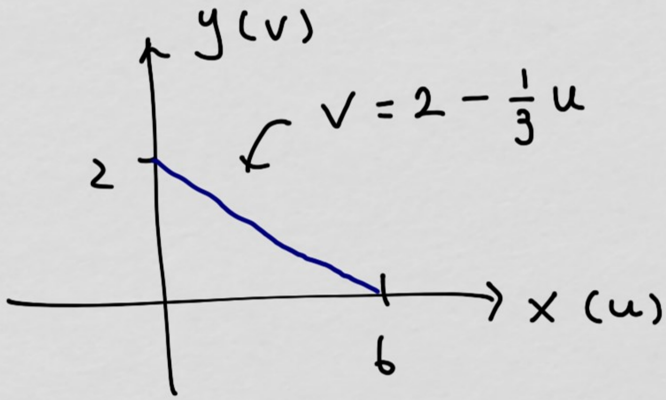
$dS = |\vec{r}_u \times \vec{r}_v| \, dA$
↳ $du\,dv$ or $dv\,du$

parametrize surface : $\vec{r}(u,v)$

let $u = x$, $v = y$, then $\underbrace{z = 3 - \frac{1}{2}x - \frac{3}{2}y}_{\text{from } x+3y+2z=6}$

$\vec{r}(u,v) = \langle u, v, 3 - \frac{1}{2}u - \frac{3}{2}v \rangle$

need bounds of u and v



$$0 \le u \le 6$$

$$0 \le v \le 2 - \frac{1}{3}u$$

$$\vec{r}(u,v) = \left\langle u,\ v,\ 3 - \frac{1}{2}u - \frac{3}{2}v \right\rangle$$

$$dS = |\vec{r}_u \times \vec{r}_v|\, dA$$

$$\vec{r}_u = \left\langle 1,\ 0,\ -\frac{1}{2} \right\rangle$$

$$\vec{r}_v = \left\langle 0,\ 1,\ -\frac{3}{2} \right\rangle$$

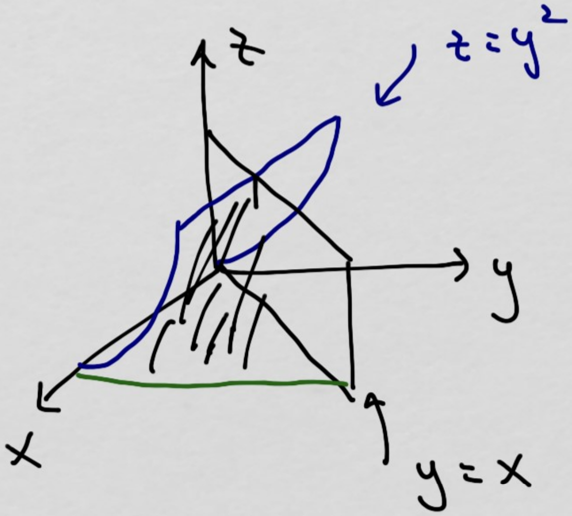
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -3/2 \end{vmatrix} = \left\langle \frac{1}{2},\ \frac{3}{2},\ 1 \right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\frac{1}{4} + \frac{9}{4} + 1} = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2} \qquad dS = \frac{\sqrt{14}}{2}\, dv\, du$$

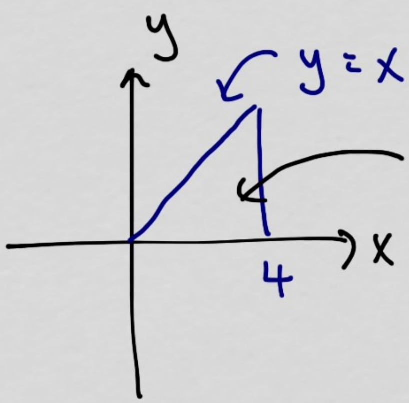
$$S = \int_0^6 \int_0^{2-\frac{1}{3}u} \frac{\sqrt{14}}{2}\, dv\, du = \cdots = 3\sqrt{14}$$

A solid region in the first octant is bounded by the surfaces $z = y^2$, $y = x$, $y = 0$, $z = 0$ and $x = 4$. The volume of the region is

A. 64 B. $\frac{64}{3}$ C. $\frac{32}{3}$ D. 32 E. $\frac{16}{3}$.



volume below the parabolic sheet $z = y^2$, above xy-plane, bounded by $y = x$



above: bounded by $z = y^2$ ("ceiling")

floor: $0 \le x \le 4$

$0 \le y \le x$

ceiling: $0 \le z \le y^2$

$$\text{volume} = \int_0^4 \int_0^x \int_0^{y^2} \underbrace{dz\,dy\,dx}_{dV}$$

$$= \int_0^4 \int_0^x y^2\,dy\,dx = \int_0^4 \frac{1}{3} y^3 \Big|_0^x dx$$

$$= \int_0^4 \frac{1}{3} x^3\,dx = \frac{1}{12} x^4 \Big|_0^4 = \frac{256}{12} = \frac{64}{3}$$

An object occupies the region bounded above by the sphere $x^2+y^2+z^2=32$ and below by the upper nappe of the cone $z^2=x^2+y^2$. The mass density at any point of the object is equal to its distance from the xy plane. Set up a triple integral in rectangular coordinates for the total mass m of the object.

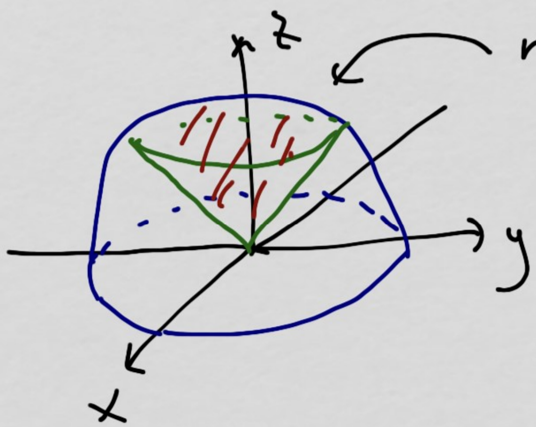
A. $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{-\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\,dz\,dy\,dx$

B. $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\,dz\,dy\,dx$

C. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\,dz\,dy\,dx$

D. $\int_{0}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\,dz\,dy\,dx$

E. $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} xy\,dz\,dy\,dx.$



mass of the icecream cone

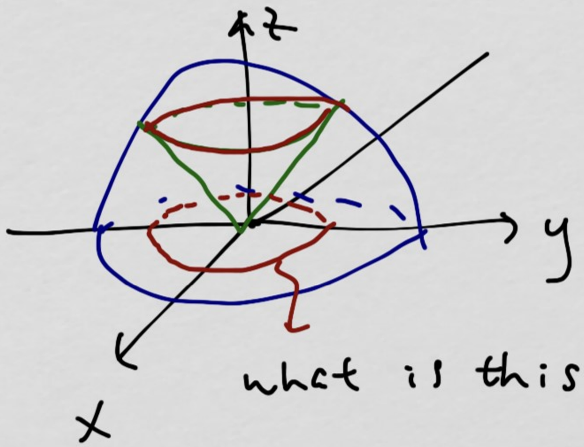
density : $\underbrace{\text{distance from } xy\text{-plane}}_{z}$

mass $= \iiint_V z\,dV$

all five choices are Cartesian, $dz\,\underbrace{dy\,dx}$

height

start w/ projection onto xy-plane



what is this?

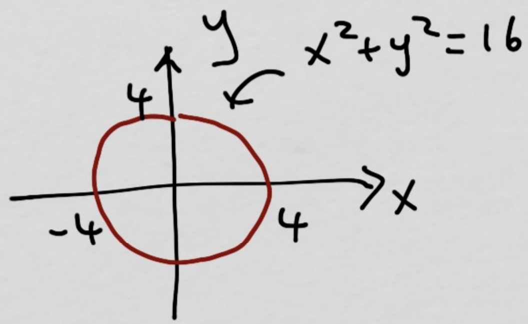
circle, but what radius?

intersection of $x^2+y^2+z^2=32$ and

$$z^2=x^2+y^2$$

$$x^2+y^2+x^2+y^2=32$$

$x^2+y^2=16 \longrightarrow$ circle of radius 4



$$-4 \le x \le 4$$

$$-\sqrt{16-x^2} \le y \le \sqrt{16-x^2}$$

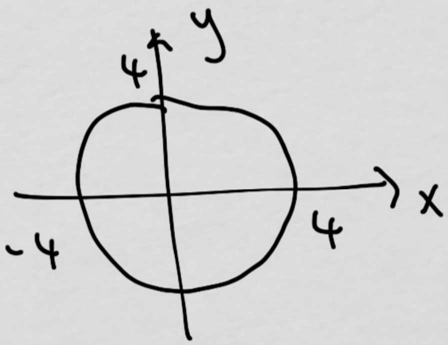
$$\sqrt{x^2+y^2} \le z \le \sqrt{32-x^2-y^2}$$

↑ cone

$$\text{mass} = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z \, dz\, dy\, dx$$

↑ density

in cylindrical ?



$$0 \le r \le 4$$

$$0 \le \theta \le 2\pi$$

$$r \le z \le \sqrt{32 - r^2}$$

$$\sqrt{x^2 + y^2} = \sqrt{r^2}$$

$$= r$$

$$\text{mass} = \int_0^{2\pi} \int_0^4 \int_r^{\sqrt{32 - r^2}} z \, r \, dz \, dr \, d\theta$$

don't forget this r in dV in cylindrical

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Determine which of the vector fields below are conservative, i. e. $\vec{F} = \text{grad } f$ for some function f.

1. $\vec{F}(x,y) = (xy^2+x)\vec{i} + (x^2y - y^2)\vec{j}$.
2. $\vec{F}(x,y) = \frac{x}{y}\vec{i} + \frac{y}{x}\vec{j}$.
3. $\vec{F}(x,y,z) = ye^z\,\vec{i} + (xe^z + e^y)\vec{j} + (xy+1)e^z\,\vec{k}$.

A. 1 and 2 B. 1 and 3 C. 2 and 3 D. 1 only E. all three

for $\vec{F} = \langle f, g \rangle$ conservative if $g_x = f_y$

for $\vec{F} = \langle f, g, h \rangle$ conservative if $\left.\begin{matrix} f_z = h_x \\ f_y = g_x \\ g_z = h_y \end{matrix}\right\}$ ALL have to be true

or more simply: if $\text{curl}\,\vec{F} = \vec{0}$

1. $\vec{F} = \underbrace{(xy^2+x)}_{f}\vec{i} + \underbrace{(x^2y - y^2)}_{g}\vec{j}$ is $\frac{\partial}{\partial x}(x^2y - y^2) = \frac{\partial}{\partial y}(xy^2+x)$?

$2xy = 2xy$ yes,

1) is conservative

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2. $\vec{F} = \underbrace{\left(\frac{x}{y}\right)}_{f}\vec{i} + \underbrace{\left(\frac{y}{x}\right)}_{g}\vec{j}$ is $\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$

$-\frac{y}{x^2} = -\frac{x}{y^2}$ no

2) is NOT conservative

3. $\vec{F} = ye^z\,\vec{i} + (xe^z + e^y)\,\vec{j} + (xy+1)e^z\,\vec{k}$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z + e^y & (xy+1)e^z \end{vmatrix}$$

$$= \langle xe^z - xe^z, -(ye^z - ye^z), e^z - e^z \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$ 3) is conservative

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Evaluate the line integral

$$\int_C x\,dx + y\,dy + xy\,dz$$

where C is parametrized by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + \cos t\vec{k}$ for $-\frac{\pi}{2} \le t \le 0$.

A. 1 B. -1 C. $\frac{1}{3}$ D. $-\frac{1}{3}$ E. 0

$\vec{r}(t) = \langle \cos t, \sin t, \cos t \rangle \qquad -\frac{\pi}{2} \le t \le 0$

$\quad x \quad y \quad z$

$x = \cos t \rightarrow dx = -\sin t\,dt$

$y = \sin t \rightarrow dy = \cos t\,dt$

$z = \cos t \rightarrow dz = -\sin t\,dt$

$$\int_C x\,dx + y\,dy + xy\,dz = \int_{-\frac{\pi}{2}}^{0} (\cos t)(-\sin t\,dt) + (\sin t)(\cos t)dt + (\cos t)(\sin t)(-\sin t)dt$$

$$= \int_{-\frac{\pi}{2}}^{0} \left(-\cos t \sin t + \sin t \cos t - \cos t \sin^2 t\right) dt$$

$$= \int_{-\frac{\pi}{2}}^{0} -\cos t \sin^2 t \, dt$$

$u = \sin t$

$du = \cos t \, dt$

$$= \int_{-1}^{0} -u^2 du = -\frac{1}{3}u^3 \Big|_{-1}^{0} = -\frac{1}{3}$$

Are the following statements true or false?

1. The line integral $\int_C (x^3+2xy)dx+(x^2-y^2)dy$ is independent of path in the xy-plane.
2. $\int_C (x^3+2xy)dx+(x^2-y^2)dy=0$ for every closed oriented curve C in the xy-plane.
3. There is a function $f(x,y)$ defined in the xy-plane, such that grad $f(x,y)=(x^3+2xy)\vec{i}+(x^2-y^2)\vec{j}$.

A. all three are false
B. 1 and 2 are false, 3 is true
C. 1 and 2 are true, 3 is false
D. 1 is true, 2 and 3 are false
E. all three are true

$$\int_C (x^3+2xy)\,dx+(x^2-y^2)\,dy=\int_C \underbrace{\langle x^3+2xy,\ x^2-y^2\rangle}_{\vec{F}}\cdot\underbrace{\langle dx,dy\rangle}_{d\vec{r}}$$

if $\vec{r}=\langle x,y\rangle$

line integral.

close loop $=0$ if $\vec{F}$ is conservative

independent of path if $\vec{F}$ is conservative

if $\vec{F}$ conservative, then $\vec{F}=\vec{\nabla}f$

so, is $\langle x^3+2xy,\ x^2-y^2\rangle$ conservative? $\frac{\partial}{\partial x}(x^2-y^2)=\frac{\partial}{\partial y}(x^3+2xy)$

$2x=2x$ yes

find f such that $\vec{\nabla} f = \langle x^3+2xy,\ x^2-y^2 \rangle$

$$= \langle f_x,\ f_y \rangle$$

$$f_x = x^3+2xy$$

$$f_y = x^2-y^2$$

$$f = \int (x^3+2xy)\,dx = \frac{1}{4}x^4 + x^2y + h(y)$$

$$\hookrightarrow f_y = x^2 + h'(y) = x^2 - y^2$$

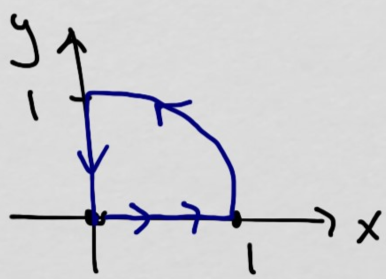
$$h'(y) = -y^2 \qquad h(y) = -\frac{1}{3}y^3 + C$$

so, $f = \frac{1}{4}x^4 + x^2y - \frac{1}{3}y^3 + C$

If C goes along the x-axis from $(0,0)$ to $(1,0)$, then along $y = \sqrt{1-x^2}$ to $(0,1)$, and then back to $(0,0)$ along the y-axis, then $\int_C xy\,dy =$

A. $-\int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$ B. $\int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$ C. $-\int_0^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx$

D. $\int_0^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx$ E. 0



$\int_C xy\,dy = ?$

one possible way: parametrize C, then evaluate the line integral

but need 3 pieces: straight segments, quarter circle

the other possible way: Green's Theorem

$\oint f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$

$\int_C \underbrace{xy}_{g}\,dy$, $f = 0$

so, $g_x - f_y = y$

$\iint_R y\,dA$

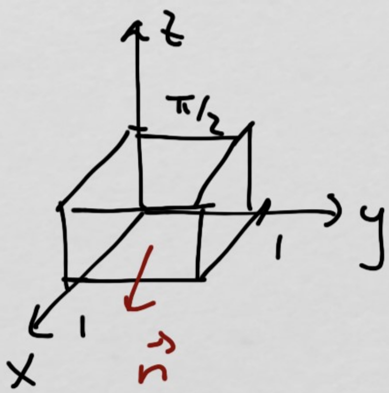
$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$

If $\vec{F}(x,y,z) = \cos z\vec{i} + \sin z\vec{j} + xy\vec{k}$, Σ is the complete boundary of the rectangular solid region bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = \frac{\pi}{2}$, and $\vec{n}$ is the outward unit normal on Σ, then $\iint_\Sigma \vec{F}\cdot\vec{n}\,dS =$

A. 0 B. $\frac{1}{2}$ C. 1 D. $\frac{\pi}{2}$ E. 2



$$\underbrace{\iint_\Sigma \vec{F}\cdot\vec{n}\,dS}_{\text{flux integral}}$$

hard way: parametrize the six surfaces then do six flux integrals

easy way: Divergence Theorem

$0 \le x \le 1$
$0 \le y \le 1$
$0 \le z \le \pi/2$

$$\iint_\Sigma \vec{F}\cdot\vec{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle \cos z, \sin z, xy\rangle$$
$$= \frac{\partial}{\partial x}(\cos z) + \frac{\partial}{\partial y}(\sin z) + \frac{\partial}{\partial z}(xy) = 0$$

$$\iiint_D 0\,dV = 0$$

Divergence Theorem for closed volume only

Use Stoke's theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

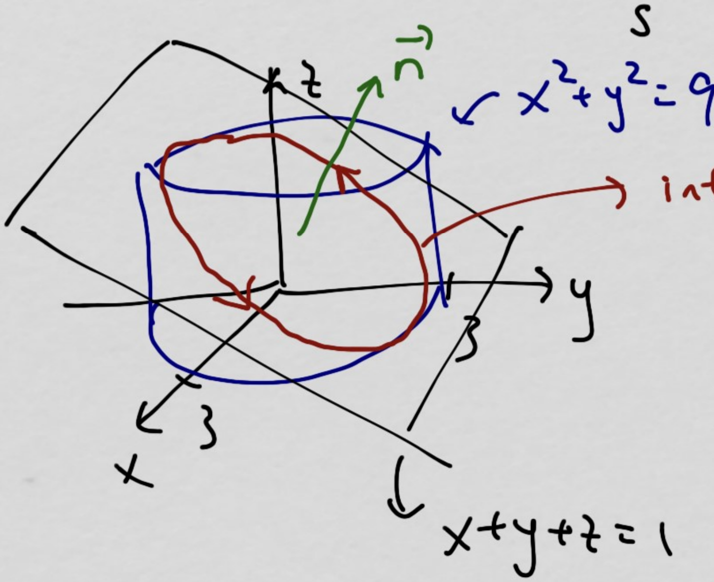
$$\vec{F}(x, y, z) = x^2 z\vec{i} + xy^2\vec{j} + z^2\vec{k},$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

A. $\frac{81\pi}{2}$ B. $\frac{\pi}{2}$ C. 1 D. $\frac{3\pi}{8}$ E. 9π

Stokes' Theorem: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$

boundary of surface



intersection of cylinder and plane

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \cdots = \langle 0, x^2, y^2 \rangle$$

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parametrize S (flat ellipse surface)

not just 3 because we want the entire ellipse

part of cylinder $x^2+y^2=9 \rightarrow x=r\cos t,\ y=r\sin t$

$0 \le t \le 2\pi$

z: comes from $x+y+z=1$

so $z=1-x-y$

$z=1-r\cos t-r\sin t$

$\vec{r}(u,v)=\langle u\cos v,\ u\sin v,\ 1-u\cos v-u\sin v\rangle$

r θ

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$\vec{r}_u=\langle \cos v,\ \sin v,\ -\cos v-\sin v\rangle$

$\vec{r}_v=\langle -u\sin v,\ u\cos v,\ u\sin v-u\cos v\rangle$

$\vec{r}_u \times \vec{r}_v = \cdots = \underbrace{\langle u,u,u\rangle}_{\text{upward? yes, because } 0\le u\le 3}$ $\quad d\vec{S}=\langle u,u,u\rangle\, du\,dv$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{3} \underbrace{\langle 0, x^2, y^2\rangle}\cdot\langle u, u, u\rangle\, du\, dv$$

$$\vec{r}(u,v) = \langle \underset{x}{u\cos v}, \underset{y}{u\sin v}, 1 - u\cos v - u\sin v\rangle$$

$$= \int_0^{2\pi}\int_0^{3} \langle 0, u^2\cos^2 v, u^2\sin^2 v\rangle\cdot\langle u, u, u\rangle\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{3} u^3\cos^2 v + u^3\sin^2 v\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{3} u^3\, du\, dv = \dots = \frac{81\pi}{2}$$