Exam Review

$$\vec{x}\,' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$$

type and stability of critical pt

homogeneous: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eigenvalues

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda) + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda = -1, \quad \lambda = -2$$

both negative : asympt. stable node
improper

opposite signs : saddle pt unstable

$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$ $\quad \lambda = -1 \pm i \quad$ find $\vec{x}(t)$

need eigenvectors, then one fundamental solution

$\underline{\lambda = -1 + i}$

solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$b = r$

$(2-i)a - b = 0$

$$a = \frac{b}{2-i} = \frac{r}{2-i}$$

choose $r = 2-i$ so $a = 1$

$$\vec{v} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

one solution: $\vec{x}_1 = e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ $\qquad e^{it} = \cos t + i \sin t$

$$= e^{-t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos t + i \sin t \\ 2\cos t + 2i \sin t - i \cos t + \sin t \end{bmatrix}$$

$$= \underbrace{e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix}}_{\vec{u}} + i \underbrace{e^{-t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}}_{\vec{v}}$$

gen solution: $c_1 \vec{u} + c_2 \vec{v}$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2(s-1)} \quad \text{find } f(t)$$

choices: $$\frac{1}{s^2(s-1)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}$$

convolution: $$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

convolution

$$\frac{1}{s^2(s-1)} = \underbrace{\boxed{\frac{1}{s^2}}}_{F}\ \underbrace{\boxed{\frac{1}{s-1}}}_{G}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t = g(t)$$

$$\int_0^t (t-\tau)e^{\tau}\,d\tau = \int_0^t \tau e^{t-\tau}\,d\tau$$

$$= \int_0^t \tau e^{-\tau} e^{t}\,d\tau = e^t \int_0^t \tau e^{-\tau}\,d\tau$$

$$\frac{1}{s^2(s-1)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}$$

multiply by $s^2(s-1)$

$$1 = As^2 + Bs(s-1) + C(s-1)$$

$$= As^2 + Bs^2 - Bs + Cs - C$$

$$1 = (A+B)s^2 + (C-B)s - C$$

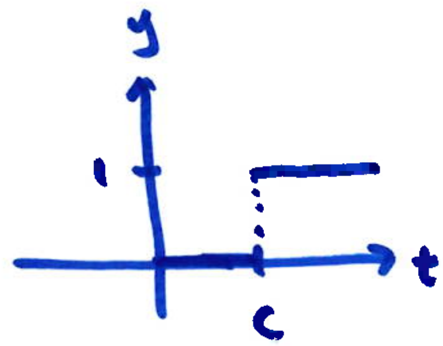
$A+B = 0$

$C-B = 0$

$-C = 1 \rightarrow C = -1, \; B = -1, \; A = 1$

$$\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \rightarrow e^t - 1 - t$$

unit step: $u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{otherwise} \end{cases}$



impulse: $\delta(t-c) = \begin{cases} \to \infty & t = c \\ 0 & \text{otherwise} \end{cases}$ $\quad \int_{-\infty}^{\infty} \delta(t-c)\,dt = 1$

$\mathcal{L}\{u(t-c)\} = \dfrac{e^{-cs}}{s}$ $\quad \mathcal{L}\{\delta(t-c)\} = e^{-cs}$

$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 2t & t \geq 1 \end{cases}$ $\quad \mathcal{L}\{f(t)\} = ?$

first write using step functions

puts at the level we want

$$f(t) = \underbrace{t}_{u(t-0)} + u(t-1)(\underbrace{-t}_{\text{resets to } 0} + 2t)$$

$$= t + u(t-1)(t)$$

shift LEFT by 1 $\to$ change t to $t+1$

$$F(s) = \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{t+1\}$$

$$= \frac{1}{s^2} + e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t^2 - 4 & 2 \le t < 3 \\ -1 & t \ge 3 \end{cases}$$

level we want

$$= u(t-2)(t^2-4) + u(t-3)\left(\underbrace{-(t^2-4)}_{\text{reset to } 0} - 1\right)$$

$x' = -y - x^2$

$y' = -x + y^2$

critical pts: $x' = 0$ AND $y' = 0$

$-y - x^2 = 0$

$-x + y^2 = 0$

$y = -x^2$

$-x + (-x^2)^2 = 0$

$x^4 - x = 0$

$x(x^3 - 1) = 0 \qquad x = 0, \quad x = 1$

$y = -x^2 \qquad y = 0, \quad y = -1$

cp: $(0,0)$, $(1,-1)$

linearize $\rightarrow$ Jacobian $\qquad x' = f(x,y) \quad y' = g(x,y)$

$$J = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} \qquad J = \begin{bmatrix} -2x & -1 \\ -1 & 2y \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ linearized sys: $\vec{x}' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}$

linearized sys: λ = ~~purely imaginary~~ $\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

linearized stability = nonlinear

unless : repeated λs

purely imaginary λs