# Final Exam

3 note cards

15-20 questions

mostly same format, but more recent might be free responce

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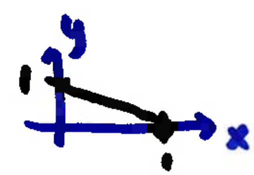
Quiz 11

$y_{tt} = y_{xx} \qquad 0<x<1$

$y_x(0,t)=0$ → left end can move up/down but not left/right

$y(1,t)=0$ → right end fixed at 0

$y(x,0)=1-x$ → initial displacement




$y_t(x,0)=0$ → initial velocity

eigenfunctions for $X$ : $X_n = \dots$

$y(x,t) = X(x)\,T(t)$

$X T'' = X'' T \qquad \dfrac{X''}{X} = \dfrac{T''}{T} = -\lambda$

$$X'' + \lambda X = 0 \qquad y_x(0,t)=0 \rightarrow X'(0)T(t)=0 \rightarrow X'(0)=0$$

$$y(1,t)=0 \rightarrow X(1)T(t)=0 \rightarrow X(1)=0$$

$$X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$X' = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$X'(0)=0 \rightarrow C_2 = 0$$

$$X = C_1 \cos(\sqrt{\lambda}x)$$

$$X(1)=0 \rightarrow C_1\cos(\sqrt{\lambda})=0 \qquad C_1 \neq 0$$

$$\cos(\sqrt{\lambda})=0$$

$$\sqrt{\lambda} = \frac{(2n-1)}{2}\pi \qquad n=1,2,3,\ldots$$

$$\boxed{\lambda_n = \frac{(2n-1)^2\pi^2}{4}} \qquad \boxed{X_n = \cos\left(\frac{(2n-1)\pi}{2}x\right)}$$

## Repeated Eigenvalues

$$\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 9 \\ 0 & -1 & 5 \end{bmatrix} \vec{x}$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & -1-\lambda & 9 \\ 0 & -1 & 5-\lambda \end{vmatrix} = 0 = (2-\lambda) \begin{vmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{vmatrix} = 0$$

$$= (2-\lambda)\left[(-1-\lambda)(5-\lambda)+9\right] = 0$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 4) = 0$$

$$= (2-\lambda)(\lambda-2)^2 = 0 \qquad \lambda = 2, 2, 2$$

$\underline{\lambda = 2}$ Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -3 & 9 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

3rd col no leading 1 → free $r$

2nd row → first element is 0

3rd row → $-x_2 + 3x_3 = 0$

$-x_2 + 3r = 0 \qquad x_2 = 3r \qquad$ let $r = 1$

missing two eigenvectors

generalized eigenvectors: $(A-\lambda I)\vec{u} = \vec{v}$ "real" eigenvector

$(A-\lambda I)^2 \vec{u} = \vec{0}$

# of missings

$(A-\lambda I)^{2+1} = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v}_3 = \vec{0}$$

choose $\vec{v}_3$ so that $(A-\lambda I)\vec{v}_3 \neq \vec{0}$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

step down: $(A-\lambda I)\vec{v}_3 = \vec{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -3 & 9 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

solutions: $\vec{x}_1 = e^{2t}\begin{bmatrix}0\\3\\1\end{bmatrix}$, $\vec{x}_2 = e^{2t}\left\{t\begin{bmatrix}0\\3\\1\end{bmatrix}+\begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}$

$\vec{x}_3 = e^{2t}\left\{\frac{1}{3}t^2\begin{bmatrix}0\\3\\1\end{bmatrix}+t\begin{bmatrix}0\\-1\\0\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$$\boxed{\text{Euler identity}: \; e^{ix} = \cos(x) + i\sin(x)}$$

$\vec{x}\,' = \begin{bmatrix}-2 & 6\\-3 & 4\end{bmatrix}\vec{x}$ $\qquad \lambda = 1 \pm 3i$

$\qquad\qquad\qquad\qquad\qquad \vec{v} = \begin{bmatrix}2\\1\pm i\end{bmatrix}$

form solution w/ one pair $\lambda = 1+3i$, $\vec{v} = \begin{bmatrix}2\\1+i\end{bmatrix}$

$$\vec{x} = e^{(1+3i)t}\begin{bmatrix}2\\1+i\end{bmatrix}$$

$$= e^{t}e^{i(3t)}\begin{bmatrix}2\\1+i\end{bmatrix} = e^{t}(\cos 3t + i\sin 3t)\begin{bmatrix}2\\1+i\end{bmatrix}$$

$$= e^{t}\left\{\begin{bmatrix}2\cos 3t\\ \cos 3t - \sin 3t\end{bmatrix} + i\begin{bmatrix}2\sin 3t\\ \cos 3t + \sin 3t\end{bmatrix}\right\}$$

gen. solution: $\vec{x} = c_1 e^{t}\begin{bmatrix}2\cos 3t\\ \cos 3t - \sin 3t\end{bmatrix} + c_2 e^{t}\begin{bmatrix}2\sin 3t\\ \cos 3t + \sin 3t\end{bmatrix}$

## Jacobian Matrix

$$\frac{dx}{dt} = f(x,y) \qquad\qquad J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

$$\frac{dy}{dt} = g(x,y)$$

evaluate at critical pt to linearize

$x' = x - 2y$
$y' = 4x - x^3$

nonlinear.
~~to~~ linearize about critical pts

cp: $x' = 0$ AND $y' = 0$

$x - 2y = 0 \rightarrow y = \frac{1}{2}x$

$4x - x^3 = 0 \rightarrow x(4 - x^2) = 0 \rightarrow x = 0,\ x = 2,\ x = -2$

$y = 0,\ y = 1,\ y = -1$

$(0,0),\ (2,1),\ (-2,-1)$

$$J = \begin{bmatrix} 1 & -2 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

near (2,1), the nonlinear system behaves like

$$\vec{x}' = \begin{bmatrix} 1 & -2 \\ -8 & 0 \end{bmatrix} \vec{x} \rightarrow \text{stability?}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ -8 & -\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(-\lambda) - 16 = 0$$

$$\lambda^2 - \lambda - 16 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+64}}{2}$$ two opposite signs $\rightarrow$ saddle pt

two same sign $\rightarrow$ proper or improper node

both positive $\rightarrow$ source

both negative $\rightarrow$ sink

improper is more common

proper $\rightarrow$ repeated $\lambda$ AND complete (enough eigenvectors)

complex $\rightarrow$ spirals