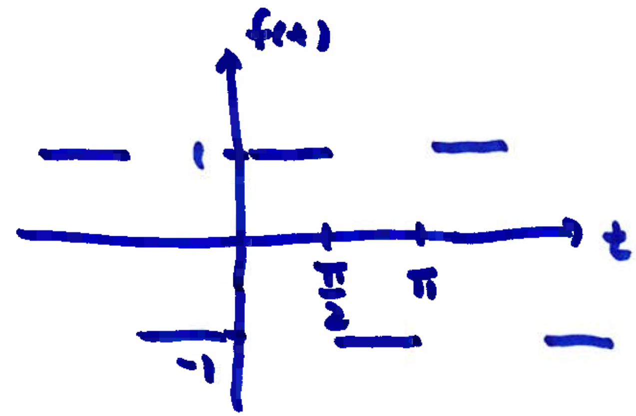
## Fourier Series

L : half period

$$f(t) = \begin{cases} 1 & 0 \le t < \pi/2 \\ -1 & \frac{\pi}{2} \le t < \pi \end{cases}$$

period $\pi$




$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$a_0 = \frac{1}{\pi/2}\int_{-\pi}^{\pi} f(t)\,dt = \frac{2}{\pi}\int_{-\pi/2}^{0} +1\,dt + \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} -1 \cdot dt = 0$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{0} \cos\left(\frac{n\pi t}{\pi/2}\right)dt + \frac{2}{\pi}\int_{0}^{\pi/2} -\cos\left(\frac{n\pi t}{\pi/2}\right)dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{0} \cos(2nt)\,dt - \frac{2}{\pi}\int_{0}^{\pi/2} \cos(2nt)\,dt$$

$$= \frac{2}{\pi}\cdot\frac{1}{2n}\sin(2nt)\Big|_{-\pi/2}^{0} - \frac{2}{\pi}\cdot\frac{1}{2n}\sin(2nt)\Big|_{0}^{\pi/2} = 0$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{0} \sin(2nt)\,dt - \frac{2}{\pi}\int_{0}^{\pi/2} \sin(2nt)\,dt$$

$$= -\frac{2}{\pi}\cdot\frac{1}{2n}\cos(2nt)\Big|_{-\pi/2}^{0} + \frac{2}{\pi}\cdot\frac{1}{2n}\cos(2nt)\Big|_{0}^{\pi/2}$$

$$= -\frac{1}{n\pi}\left(1-\cos(n\pi)\right) + \frac{1}{n\pi}\left(\cos(n\pi)-1\right)$$

$$= \frac{2}{n\pi}\left[\cos(n\pi)-1\right] = -\frac{2}{n\pi}\left[1-\cos(n\pi)\right] \qquad n=1,2,3,\ldots$$

$$f(t) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left[1-\cos(n\pi)\right]\sin(2nt)$$

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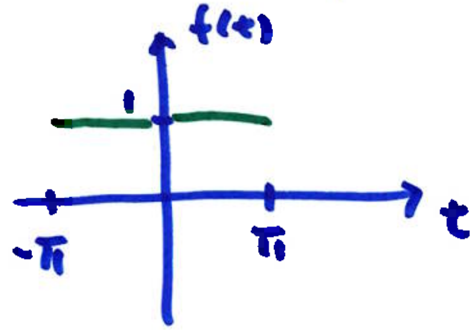
$f(t) = 1$ period $2\pi$ even



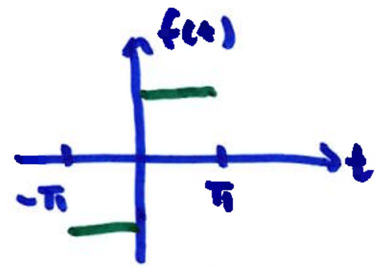

$b_n = 0$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 1\,dt = 2$$

$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nt)\,dt = 0$ Fourier $\frac{1}{2}(a_0) = 1$

odd extension




$a_n = 0$ $b_n = \frac{2}{L}\int_0^{L} f(t) \sin\left(\frac{n\pi t}{L}\right)dt$

$= \frac{2}{\pi}\int_0^{\pi} \sin(nt)\,dt = -\frac{2}{n\pi}\cos(nt)\Big|_0^{\pi}$

$= -\frac{2}{n\pi}\left[\cos(n\pi) - 1\right]$ $n = 1, 2, 3, \ldots$

$= -\frac{2}{n\pi}\left[(-1)^n - 1\right]$

Sine series : $\sum_{n=1}^{\infty} -\frac{2}{n\pi}\left[(-1)^n - 1\right]\sin(nt)$

$x'' + x = f(t) \qquad f(t) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left[(-1)^n - 1\right]\sin(nt)$

$x'' + x = 0 \rightarrow x(t) = c_1\cos(t) + c_2\sin(t) \qquad n=1 \rightarrow$ resonance

series solution: $x = \sum_{n=1}^{\infty} A_n \sin(nt)$ sine series

$$x' = \sum_{n=1}^{\infty} n A_n \cos(nt)$$

$$x'' = \sum_{n=1}^{\infty} -n^2 A_n \sin(nt)$$

$$\sum_{n=1}^{\infty} -n^2 A_n \sin(nt) + \sum_{n=1}^{\infty} A_n \sin(nt) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left[(-1)^n - 1\right]\sin(nt)$$

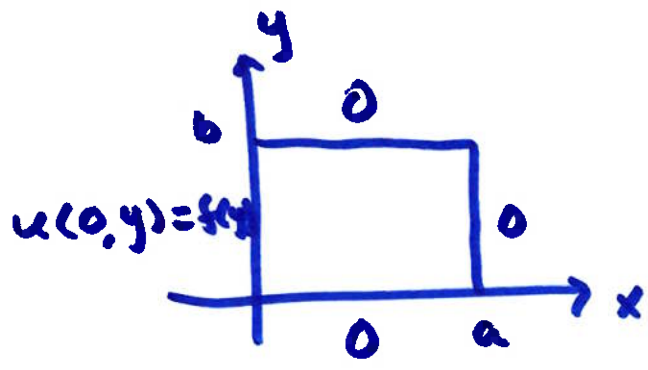
equating coefficients

$$-n^2 A_n + A_n = -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$A_n = -\frac{2}{n\pi(1-n^2)}\left[(-1)^n - 1\right]$$

if $n=1$ magnitude $\rightarrow \infty$

Laplace $u_{xx} + u_{yy} = 0$ $\quad 0 < x < a \quad 0 < y < b$




$u(x,0) = 0$

$u(x,b) = 0$

$u(a,y) = 0$

$u(0,y) = f(y)$

$u = XY \qquad u_{xx} = X''Y \qquad u_{yy} = XY''$

$X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

$u(x,0)=0 \to Y(0)=0$
$u(x,b)=0 \to Y(b)=0$ } solve Y first
$u(a,y)=0 \to X(a)=0$

$$-\frac{Y''}{Y} = c$$

expect sine and cosine in solution of Y

$c > 0 \to c = \lambda^2$

$Y'' + cY = 0 \to Y'' + \lambda^2 Y = 0$

$Y = C_1 \cos(\sqrt{\lambda}\, y) + C_2 \sin(\sqrt{\lambda}\, y) = 0$

$Y(0) = 0 \rightarrow C_1 = 0$

$Y(b) = 0 \rightarrow C_2 \sin(\lambda b) = 0 \qquad \lambda b = n\pi \qquad \lambda = \frac{n\pi}{b}$

$Y_n = \sin\left(\frac{n\pi y}{b}\right)$

$\frac{X''}{X} = C = \lambda^2 = \frac{n^2\pi^2}{b^2} \qquad X(a) = 0$

$X'' - \frac{n^2\pi^2}{b^2} X = 0$

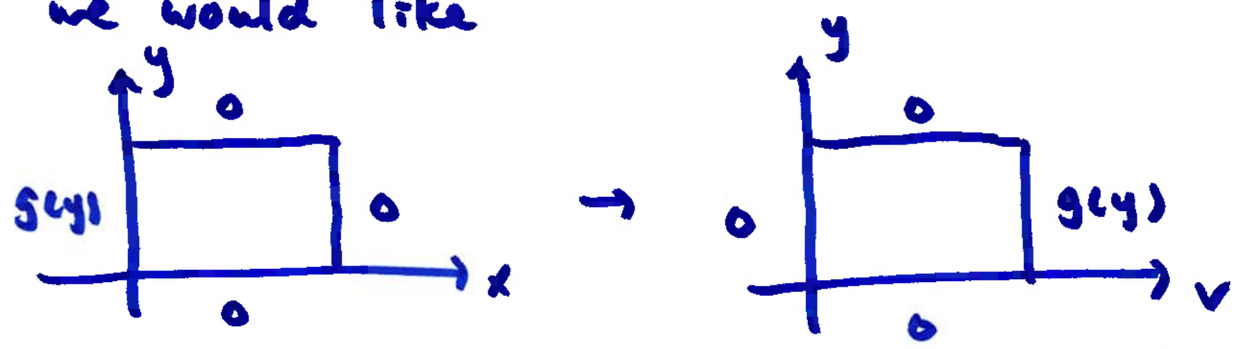
$X = C_1 \cosh\left(\frac{n\pi x}{b}\right) + C_2 \sinh\left(\frac{n\pi x}{b}\right)$

$X(a) = 0 \rightarrow C_1 \cosh\left(\frac{n\pi a}{b}\right) + C_2 \sinh\left(\frac{n\pi a}{b}\right)$

neither $C_1$ nor $C_2$ can be eliminated

solution: change of variable

we would like




reflection about y-axis : change $x$ to $-x$

translate right by $a$ : add $a$

so, $v = a - x$

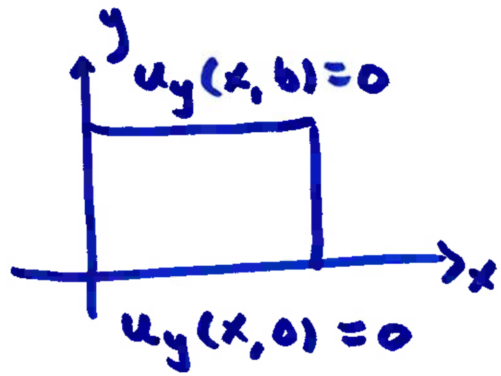
$$\frac{X''(v)}{X(v)} = \frac{n^2\pi^2}{b^2} \qquad X(x=a)=0 \rightarrow X(v=0)=0$$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0$$

$$X = c_1 \cosh\left(\frac{n\pi v}{b}\right) + c_2 \sinh\left(\frac{n\pi v}{b}\right)$$

$$X(v=0) = 0 \rightarrow c_1 = 0 \qquad X_n = \sinh\left(\frac{n\pi v}{b}\right)$$

$$X_n = \sinh\left(\frac{n\pi(a-x)}{b}\right)$$




BC's in terms $Y \rightarrow$ $Y'(0)=0$
$Y'(b)=0$

**MA 30300-122** **Final Exam** **Spring 2024**

**Name:** ______________________________

Instructions:

1. This is a two-hour exam.
2. There are 15 problems on this exam. Each problem is worth 10 points.
3. The last two pages contain some potentially useless formulas and a table of Laplace transforms. You may tear out those pages.
4. No calculators are allowed. You may have three $4 \times 6$ note cards.
5. Please put your phone away during the exam.
6. To earn full credit please provide relevant work, justification, or reasoning to explain how the answer is obtained. You may re-use results or calculations from other problems on the exam. Please be sure to cite your own work.
7. Circle one and only one choice for each problem.
8. 50% partial credit may be given for correct steps leading to the correct solution.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes (other than the three ~~one~~ 4x6 index card), books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

**Signature:** ______________________________

## Potentially Useless Information

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

$$\int t\sin(at)dt = \frac{\sin(at)}{a^2} - \frac{t\cos(at)}{a} + C$$

$$\int t\cos(at)dt = \frac{\cos(at)}{a^2} + \frac{t\sin(at)}{a} + C$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right) dt = \begin{cases} 0 & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt = 0$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

| Function | Transform | Function | Transform |
|---|---|---|---|
| $f(t)$ | $F(s)$ | $e^{at}$ | $\dfrac{1}{s-a}$ |
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ | $t^n e^{at}$ | $\dfrac{n!}{(s-a)^{n+1}}$ |
| $f'(t)$ | $sF(s)-f(0)$ | $\cos kt$ | $\dfrac{s}{s^2+k^2}$ |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ | $\sin kt$ | $\dfrac{k}{s^2+k^2}$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$ | $\cosh kt$ | $\dfrac{s}{s^2-k^2}$ |
| $\int_0^t f(\tau)\,d\tau$ | $\dfrac{F(s)}{s}$ | $\sinh kt$ | $\dfrac{k}{s^2-k^2}$ |
| $e^{at}f(t)$ | $F(s-a)$ | $e^{at}\cos kt$ | $\dfrac{s-a}{(s-a)^2+k^2}$ |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ | $e^{at}\sin kt$ | $\dfrac{k}{(s-a)^2+k^2}$ |
| $\int_0^t f(\tau)g(t-\tau)\,d\tau$ | $F(s)G(s)$ | $\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$ | $\dfrac{1}{(s^2+k^2)^2}$ |
| $tf(t)$ | $-F'(s)$ | $\dfrac{t}{2k}\sin kt$ | $\dfrac{s}{(s^2+k^2)^2}$ |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ | $\dfrac{1}{2k}(\sin kt + kt\cos kt)$ | $\dfrac{s^2}{(s^2+k^2)^2}$ |
| $\dfrac{f(t)}{t}$ | $\int_s^\infty F(\sigma)\,d\sigma$ | $u(t-a)$ | $\dfrac{e^{-as}}{s}$ |
| $f(t)$, period $p$ | $\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$ | $\delta(t-a)$ | $e^{-as}$ |
| $1$ | $\dfrac{1}{s}$ | $(-1)^{[\![t/a]\!]}$ (square wave) | $\dfrac{1}{s}\tanh\dfrac{as}{2}$ |
| $t$ | $\dfrac{1}{s^2}$ | $\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase) | $\dfrac{e^{-as}}{s(1-e^{-as})}$ |
| $t^n$ | $\dfrac{n!}{s^{n+1}}$ | | |
| $\dfrac{1}{\sqrt{\pi t}}$ | $\dfrac{1}{\sqrt{s}}$ | | |
| $t^a$ | $\dfrac{\Gamma(a+1)}{s^{a+1}}$ | | |