

Laplace Transform

$$y'' + y = 3 \quad y(0) = 2 \quad y'(0) = 1$$

$$\mathcal{L}\{y'\} = sY - y(0)$$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{3\}$$

$$s^2Y - sy(0) - y'(0) + Y = \frac{3}{s} \quad \text{find } Y, \text{ then } y = \mathcal{L}^{-1}\{Y\}$$

$$(s^2 + 1)Y = 2s + 1 + \frac{3}{s}$$

$$Y = \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{3}{s(s^2+1)}$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{3}{s} - \frac{3s}{s^2+1}$$

$$= \frac{3}{s} + \frac{1}{s^2+1} - \frac{s}{s^2+1}$$

$$y = 3 + \sin(t) - \cos(t)$$

$$\frac{3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$3 = A(s^2+1) + (Bs+C)(s)$$

$$3 = (A+B)s^2 + Cs + A$$

$$A+B=0$$

$$C=0$$

$$A=3$$

$$B=-3$$

$$\text{convolution: } f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau$$

$$\mathcal{L}\{f(t) * g(t)\} = F(s) G(s)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} \frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1}\right\}$$

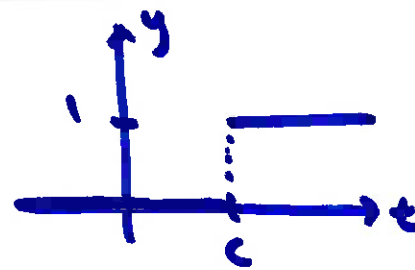
$$\mathcal{L}^{-1}\left\{\overset{F}{\boxed{\frac{1}{s-1}}} \overset{G}{\boxed{\frac{1}{s+1}}}\right\} \quad f(t) = e^t \quad g(t) = e^{-t}$$

$$= \int_0^t e^{t-\tau} e^{-\tau} d\tau = \int_0^t e^t e^{-2\tau} d\tau = e^t \int_0^t e^{-2\tau} d\tau$$

$$= e^t \left(-\frac{1}{2} e^{-2\tau}\right) \Big|_0^t = -\frac{1}{2} e^t (e^{-2t} - 1) = \frac{1}{2} e^t - \frac{1}{2} e^{-t}$$

$$= \sinh(t)$$

$$u(t-c) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$



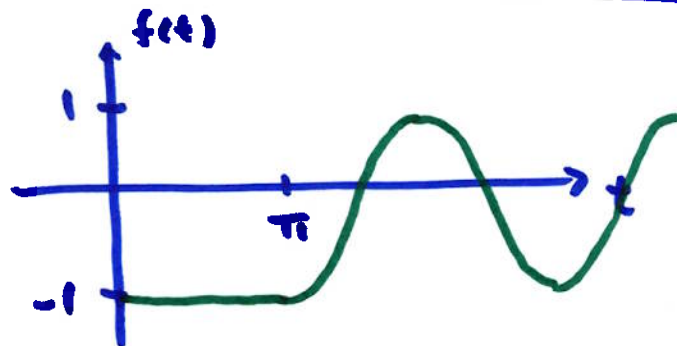
$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s)$$

we don't transform the function
shift LEFT by c as-is
then transform

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c) \quad \text{shift RIGHT after inverte transform}$$

$$f(t) = \begin{cases} -1 & 0 < t < \pi \\ \cos(t) & t \geq \pi \end{cases} \quad (t > 0)$$



write $f(t)$ in terms of unit step

$$f(t) = -1 + u(t-\pi) [1 + \cos(t)]$$

$$F(s) = \mathcal{L}\{-1\} + \mathcal{L}\{u(t-\pi) [1 + \cos(t)]\}$$

Shift LEFT by $\pi : t \rightarrow t+\pi$
so we transform $1 + \cos(t+\pi)$

$$= -\frac{1}{s} + e^{-\pi s} \mathcal{L}\{1 + \cos(t+\pi)\}$$

$$= -\frac{1}{s} + e^{-\pi s} \mathcal{L}\{1 - \cos(t)\}$$

$$\cos(t+\pi) = -\cos(t)$$

$$= -\frac{1}{s} + e^{-\pi s} \left(\frac{1}{s} - \frac{s}{s^2+1} \right)$$

$$F(s) = \frac{e^{-2s}}{s^2-2s+2}$$

$$f(t) = ?$$

$$= e^{-2s} \boxed{\frac{1}{s^2-2s+2}}$$

Inv. transform
then shift t RIGHT by 2 : $t \rightarrow t-2$

$$\frac{1}{s^2-2s+2} = \frac{1}{(s^2-2s+1)+1} = \frac{1}{(s-1)^2+1} \rightarrow e^t \sin(t)$$

$$f(t) = u(t-2) [e^{t-2} \sin(t-2)]$$

$$\text{impulse : } \delta(t-c) = \begin{cases} \rightarrow \infty & t=c \\ 0 & t \neq c \end{cases} \quad \int_{-\infty}^{\infty} \delta(t-c) dt = 1$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

$$y'' + y = \delta(t-\pi) + \delta(t-2\pi) \quad y(0)=0, y'(0)=0$$

$$s^2 Y - \cancel{s y(0)} - \cancel{y'(0)} + Y = e^{-\pi s} + e^{-2\pi s}$$

$$(s^2+1)Y = e^{-\pi s} + e^{-2\pi s}$$

$$Y = e^{-\pi s} \frac{1}{s^2+1} + e^{-2\pi s} \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$y = u(t-\pi) \sin(t-\pi) + u(t-2\pi) \sin(t-2\pi) = \begin{cases} 0 & 0 < t < \pi \\ -\sin(t) & \pi < t < 2\pi \\ 0 & t > 2\pi \end{cases}$$