5.2 The Eigenvalue Method for Linear Systems

Systems of first-order ODEs

for example, $x_1' = x_1 + 2x_2$ solution: $x_1(t) = ?$

$x_2' = 2x_1 + x_2$ $x_2(t) = ?$

matrix form: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\vec{x}\,' = A\vec{x}$

coefficient matrix

Solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$\left.\begin{matrix} \lambda_1, \vec{v}_1 \\ \lambda_2, \vec{v}_2 \end{matrix}\right\}$ eigenvalue and eigenvector pairs of A

n pairs for n×n A

example : $x_1' = x_1 + 2x_2$

$x_2' = 2x_1 + x_2$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

identity matrix

eigenvalues : solve $\det(A - \lambda I) = 0$ for λ

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

$(1-\lambda)^2 = 4$

$(1-\lambda) = 2$ or $1-\lambda = -2$

$\lambda_1 = -1, \quad \lambda_2 = 3$

find corresponding eigenvectors

solve $(A-\lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$\underline{\lambda_1 = -1}$ $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ b is free because no pivot in col 2

$b = r$, row 1: $a + b = 0 \rightarrow a = -b = -r$

so, $\vec{v} = \begin{bmatrix} -r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ r is arbitrary

let $r = 1$

$$\boxed{\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_1 = -1}$$

Similarly, $\boxed{\lambda_2 = 3, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

general solution: $\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} -c_1 e^{-t} + c_2 e^{3t} \\ c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}$$

c_1, c_2 based on initial conditions

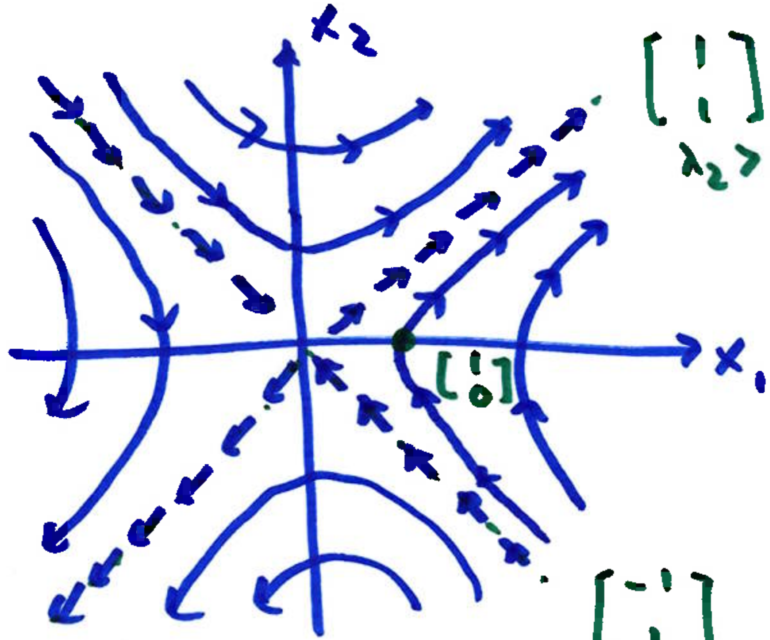
for example, $\vec{x}_1(0) = 1$, $\vec{x}_2(0) = 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \qquad c_1 = -1/2, \quad c_2 = 1/2$$

particular solution: $\vec{x}(t) = \begin{bmatrix} 1/2\, e^{-t} + 1/2\, e^{3t} \\ -1/2\, e^{-t} + 1/2\, e^{3t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Direction Field

$$\vec{x}(t) = c_1 e^{-t} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}_1} + c_2 e^{3t} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_2}$$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 > 0$ away from origin

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_1 < 0$ toward origin

Complex Eigenvalues

example $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

$\det(A - \lambda I) = 0 \qquad \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 + 1 = 0$

$(1-\lambda)^2 = -1 \qquad 1-\lambda = i \quad \text{or} \quad 1-\lambda = -i$

$\lambda_1 = 1-i \qquad \lambda_2 = 1+i$

complex : always conjugate pairs

eigenvectors :

$\underline{\lambda_1 = 1-i}$ $\quad A - \lambda I = \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$

so, $\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

eigenvectors also conjugate pairs

$\lambda_2 = 1+i, \quad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$\lambda_1 = 1-i \qquad e^{\lambda_1 t} = e^{(1-i)t} = e^t e^{-it}$

$$\boxed{\text{Euler's formula: } e^{ix} = \cos x + i \sin x}$$

$e^{-it} = \cos t - i \sin t$

$$e^{\lambda_1 t}\vec{v}_1 = e^t(\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} i\cos t + \sin t \\ \cos t - i \sin t \end{bmatrix}$$

$$= \underbrace{e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{u}} + i\, \underbrace{e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{w}}$$

general solution: $\vec{x}(t) = c_1 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

direction field : spiral

in if real part of λ is < 0

out " " " " " > 0

this example, spirals <u>out</u>

<u>example</u> $\vec{x}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \vec{x}$

solution: $c_1 e^{\lambda_1 t} \vec{v_1} + c_2 e^{\lambda_2 t} \vec{v_2} + c_3 e^{\lambda_3 t} \vec{v_3}$

eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = 0$$

cofactor expansion

$$(3-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -2 & 2 \\ 4 & -2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -2 & 3-\lambda \\ 4 & -1 \end{vmatrix} = 0$$

$\vdots$

$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \rightarrow \lambda = -1, 2, 3$, then $\vec{v_1}, \vec{v_2}, \vec{v_3}$