7.2 Transforms of Initial Value Problems

"old" way to solve $y'' - 6y' + 8y = 0 \quad y(0) = -2 \quad y'(0) = 6$

characteristic eq. $r^2 - 6r + 8 = 0$

$(r - 4)(r - 2) = 0$

$r = 2, \; r = 4$

general solution: $y = c_1 e^{2t} + c_2 e^{4t}$

$y' = 2c_1 e^{2t} + 4c_2 e^{4t}$

initial conditions: $-2 = c_1 + c_2$

$6 = 2c_1 + 4c_2$

$-4 = 2c_1 + 2c_2$

$10 = 2c_2 \rightarrow c_2 = 5$

$c_1 = -7$

$y = -7e^{2t} + 5e^{4t}$

today: solve using LT

first, given unknown $y(t)$, what is $\mathcal{L}\{y'(t)\} = ?$

assume $\mathcal{L}\{y(t)\} = Y(s)$ exists

then $\mathcal{L}\{y'\} = \int_0^\infty y' \cdot e^{-st}\,dt$

$$= \lim_{a\to\infty} \int_0^a y' \cdot e^{-st}\,dt \qquad u = e^{-st} \quad dv = y'dt$$

$$du = -se^{-st}dt \quad v = y$$

"constant"

$$= \lim_{a\to\infty} \left(ye^{-st}\Big|_0^a + \int_0^a y\,s\,e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} \left(ye^{-st}\Big|_0^a \right) + s\int_0^\infty ye^{-st}\,dt$$

$$= \left[\overset{0}{y(a)e^{-sa}} - y(0) \right] + s \underbrace{\int_0^\infty ye^{-st}\,dt}_{\mathcal{L}\{y\} = Y}$$

$$\boxed{\mathcal{L}\{y'\} = sY - y(0)}$$

likewise, $\boxed{\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)}$ $\boxed{\mathcal{L}\{y'\} = sY - y(0)}$

now, try it on $y'' - 6y' + 8y = 0 \quad y(0) = -2, \; y'(0) = 6$

LT both sides: $\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \mathcal{L}\{\overset{\nearrow 0.1}{0}\} = 0 \cdot \mathcal{L}\{1\} = 0 \cdot \frac{1}{s} = 0$

$$\Big(s^2Y - s\underset{\uparrow\,-2}{\boxed{y(0)}} - \underbrace{y'(0)}_{6}\Big) - 6\Big(sY - \underbrace{y(0)}_{-2}\Big) + 8Y = 0$$

$s^2Y + 2s - 6 - 6sY - 12 + 8Y = 0$ solve for Y

$$\underbrace{(s^2 - 6s + 8)}_{\text{char. eq.}}Y = 18 - 2s$$

$$Y = \frac{18 - 2s}{s^2 - 6s + 8}$$ now invert LT to find y

$$= \frac{18 - 2s}{(s-4)(s-2)}$$ nothing on table looks like

would be nice if $$= \frac{A}{s-4} + \frac{B}{s-2}$$ partial fractions!

$$\frac{18-2s}{(s-4)(s-2)} = \frac{A}{s-4} + \frac{B}{s-2}$$

$$18-2s = A(s-2) + B(s-4)$$

$$-2s+18 = (A+B)s + (-2A-4B)$$

$$A+B = -2$$

$$\left.\begin{aligned} -2A-4B &= 18 \\ 2A+2B &= -4 \end{aligned}\right\} -2B = +14$$

$B = -7$

$A = 5$

so, $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{5 \cdot \frac{1}{s-4} - 7 \cdot \frac{1}{s-2}\right\}$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - 7\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\boxed{y = 5e^{4t} - 7e^{2t}}$$

<u>example</u> $y'' + 4y = 4 \qquad y(0) = 1, \quad y'(0) = -2$

"old" way: solve the homogeneous part
then use undetermined coefficients
(involves guessing)
or variation of parameters (what's that?!)

LT way: transform, then solve for Y, then inverse LT

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{4\}$$

$$\left(s^2 Y - s y(0) - y'(0)\right) + 4Y = \frac{4}{s}$$

$$s^2 Y - s + 2 + 4Y = \frac{4}{s}$$

$$(s^2+4)Y = s - 2 + \frac{4}{s}$$

$$Y = \frac{s}{s^2+4} - \frac{2}{s^2+4} + \underbrace{\frac{4}{s(s^2+4)}}_{\text{partial fraction}}$$

$$= \frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{1}{s} - \frac{s}{s^2+4}$$

$Y = \frac{1}{s} - \frac{2}{s^2+4}$ so, $y = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^2+4}\right\}$

$$\boxed{y = 1 - \sin 2t}$$

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we know LT of deriv. of y

what about LT of integral? $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$

turns out $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$ where $F = \mathcal{L}\{f\}$

sometimes useful, e.g. $\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{4}{s(s^2+4)}}_{\text{partial fraction}}\Bigg\}$

or $\mathcal{L}^{-1}\left\{\frac{\overbrace{\frac{4}{s^2+4}}^{2\sin 2t}}{s}\right\} = \int_0^t 2\sin 2\tau\,d\tau$

$= -\cancel{2}\cos 2\tau\Big|_0^t = -\frac{1}{2}\cancel{\cos 2t}\ \cancel{+}\ \frac{1}{2}$

$= -\cos 2t + 1$