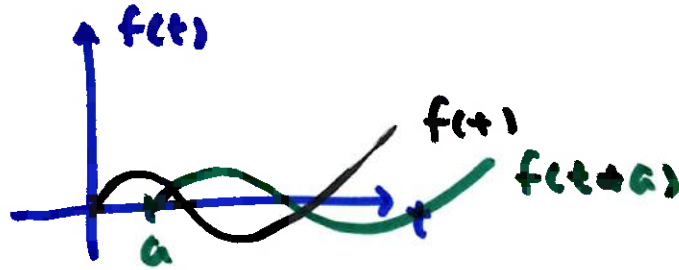


7.3 Translation of Laplace Transforms

Given $f(t)$, $f(t-a)$ translates $f(t)$ by a units



$\mathcal{L}\{f(t)\} = F(s)$ what is $F(s-a)$? unfortunately, not as in time domain (t)

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt = F(s)$$

change s to $s-a$: $F(s-a) = \int_0^{\infty} f(t) e^{-(s-a)t} dt$

$$F(s-a) = \int_0^{\infty} f(t) e^{-st} e^{at} dt$$

$$= \int_0^{\infty} (f(t) e^{at}) e^{-st} dt = \mathcal{L}\{f(t) e^{at}\} = F(s-a)$$

example $\mathcal{L}\{ \underbrace{e^t}_{a=1} \underbrace{\cos 2t}_{f(t)} \}$

$$F(s-a) = \mathcal{L}\{e^{at} f(t)\}$$

$$f(t) = \cos 2t \quad F(s) = \frac{s}{s^2+4}$$

$$F(s-a) = F(s-1) = \frac{s-1}{(s-1)^2+4}$$

$$\text{so, } \boxed{\mathcal{L}\{e^t \cos 2t\} = \frac{s-1}{(s-1)^2+4}}$$

example $\mathcal{L}^{-1}\left\{ \frac{3s+5}{\underbrace{s^2-6s+25}_{\text{sum/diff. of squares}}} \right\}$

sum/diff. of squares

$$\frac{3s+5}{s^2-6s+9+16} = \frac{3s+5}{(s-3)^2+4^2}$$

$$\frac{3(s-3)}{(s-3)^2 + 4^2} + \frac{14}{(s-3)^2 + 4^2}$$

$$= 3 \cdot \underbrace{\frac{s-3}{(s-3)^2 + 4^2}}_{\text{looks like}} + \frac{14}{4} \cdot \underbrace{\frac{4}{(s-3)^2 + 4^2}}_{\text{looks like}}$$

$$\frac{s}{s^2 + 4^2} \rightarrow \cos 4t$$

w/ s replaced by $\underbrace{s-3}_{a=3}$

$$\frac{4}{(s-3)^2 + 4^2} \rightarrow \sin 4t$$

w/ s replaced by $\underbrace{s-3}_{a=3}$

So, inverse Laplace of this gives

$$3e^{3t} \cos 4t + \frac{7}{2} e^{3t} \sin 4t$$

we could also just use table entries $\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$

$$\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$$

$$\begin{aligned}\frac{3s+5}{s^2-6s+25} &= \frac{3s}{(s-3)^2+4^2} + \frac{5}{(s-3)^2+4^2} \\&= \frac{3s-9}{(s-3)^2+4^2} + \frac{5+9}{(s-3)^2+4^2} \\&= 3 \cdot \underbrace{\frac{s-3}{(s-3)^2+4^2}}_{e^{3t}\cos 4t} + \frac{14}{4} \cdot \underbrace{\frac{4}{(s-3)^2+4^2}}_{e^{3t}\sin 4t}\end{aligned}$$

example $x'' + 4x' + 13x = te^{-t}$ $x(0) = 0, x'(0) = 2$

old way: solve homogeneous part $x'' + 4x' + 13x = 0$

$$x = C_1(\dots) + C_2(\dots)$$

then solve for C_1, C_2 using initial conditions

then nonhomogeneous part

undetermined coeffs: $x_p = Ate^{-t} + Be^{-t}$

then plug into DE to solve for A, B

Laplace way: transform, solve for $\mathcal{L}\{x\}$, then $\mathcal{L}^{-1}$ that

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 13\mathcal{L}\{x\} = \mathcal{L}\{te^{-t}\}$$

$$(s^2 \underline{X} - \cancel{s x(0)}^0 - x'(0)) + 4(s \underline{X} - \cancel{x(0)}^0) + 13 \underline{X} = \frac{1}{(s+1)^2}$$

$$(s^2 + 4s + 13) \underline{X} - 2 = \frac{1}{(s+1)^2}$$

$$\underline{X} = \frac{2}{\underbrace{s^2 + 4s + 13}_{(s+2)^2 + 3^2}} + \frac{1}{\underbrace{(s+1)^2 (s^2 + 4s + 13)}}_{\text{partial fraction expansion}}$$

partial fraction expansion

$$\underline{X} = \frac{2}{(s+2)^2 + 3^2} + \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4s+13}$$

$$= \frac{2}{(s+2)^2 + 3^2} + \frac{1}{50} \cdot \frac{s-2}{(s+2)^2 + 3^2} - \frac{1}{50} \frac{1}{s+1} + \frac{1}{10} \frac{1}{(s+1)^2}$$

$$= \frac{2}{(s+2)^2 + 3^2} + \frac{1}{50} \left[\frac{s+2}{(s+2)^2 + 3^2} - \frac{4}{(s+2)^2 + 3^2} \right] - \frac{1}{50} \frac{1}{s+1} + \frac{1}{10} \frac{1}{(s+1)^2}$$

$$x(t) = \frac{2}{3} e^{-2t} \sin(3t) + \frac{1}{50} \left[e^{-2t} \cos(3t) - \frac{4}{3} e^{-2t} \sin(3t) \right] - \frac{1}{50} e^{-t} + \frac{1}{10} t e^{-t}$$