

7.4 Derivatives, Integrals, and Multiplications of LTs

last time: $F(s-a) = \mathcal{L}\{e^{at} f(t)\}$

today: $F'(s)$, $\int_s^\infty F(\sigma) d\sigma$, $F(s)G(s)$

derivative: $F(s) = \int_0^\infty f(t) e^{-st} dt$

$$F'(s) = \frac{d}{ds} \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty f(t) \frac{d}{ds} e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-st} \cdot -t dt$$

$$= \int_0^\infty [-t f(t)] e^{-st} dt = \text{LT of } -t f(t)$$

$$\boxed{F'(s) = \mathcal{L}\{-t f(t)\}}$$

$$\text{or } \mathcal{L}^{-1}\{F'(s)\} = -t f(t) \quad \text{or } f(t) = \frac{\mathcal{L}^{-1}\{F'(s)\}}{-t}$$

Example $F(s) = \ln\left(\frac{s}{s^2-9}\right) \quad f(t) = ?$

definitely not on the Table of common transforms

$$F(s) = \ln(s) - \ln(s^2-9)$$

$$F'(s) = \underbrace{\frac{1}{s}} - \underbrace{\frac{2s}{s^2-9}}$$

we see this in the Table

$$\text{inv. transform } F'(s), \text{ then recover } f(t) = \frac{\mathcal{L}^{-1}\{F'\}}{-t}$$

$$\mathcal{L}^{-1}\{F'\} = 1 - 2 \cosh 3t$$

$$f(t) = \frac{\mathcal{L}^{-1}\{F'\}}{-t} = \boxed{-\frac{1}{t}(1 - 2 \cosh 3t)}$$

$$\sinh(t) = \frac{1}{2} (e^t - e^{-t})$$

$$\cosh(t) = \frac{1}{2} (e^t + e^{-t})$$

$$\mathcal{L} \{ \sinh t \} = \frac{1}{s^2 - 1}$$

$$= \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{1/2}{s-1} + \frac{-1/2}{s+1}$$

$$= \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1} \right\}$$

$$= \frac{1}{2} e^t - \frac{1}{2} e^{-t}$$

now $\int_s^\infty F(\sigma) d\sigma$

$$F(\sigma) = \int_0^\infty f(t) e^{-\sigma t} dt$$

$$\int_s^\infty F(\sigma) d\sigma = \int_s^\infty \int_0^\infty f(t) e^{-\sigma t} dt d\sigma$$

$$= \int_0^\infty \left(\int_s^\infty f(t) e^{-\sigma t} d\sigma \right) dt$$

$$= \int_0^\infty f(t) e^{-\sigma t} \cdot -\frac{1}{t} \Big|_s^\infty dt$$

$$= \int_0^\infty f(t) \left(\frac{e^{-st}}{t} \right) dt = \int_0^\infty \left[\frac{f(t)}{t} \right] e^{-st} dt$$

$$\boxed{\int_s^\infty F(\sigma) d\sigma = \mathcal{L} \left\{ \frac{f(t)}{t} \right\}}$$

example $\mathcal{L} \left\{ \frac{\sin t}{t} \right\}$

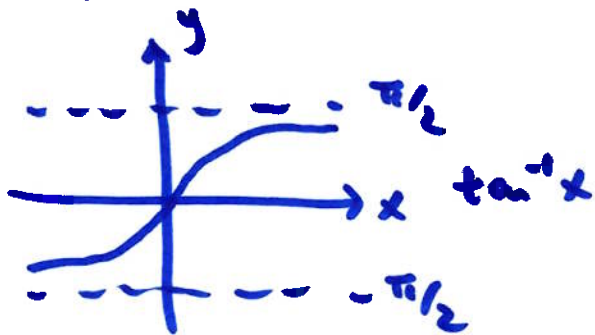
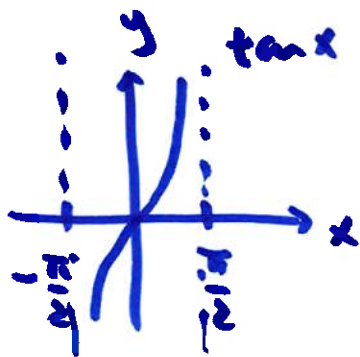
$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^{\infty} F(\sigma) d\sigma$$

$$\mathcal{L} \{ \sin t \} = \frac{1}{s^2 + 1}$$

$$\mathcal{L} \left\{ \frac{\sin t}{t} \right\} = \int_s^{\infty} \frac{1}{\sigma^2 + 1} d\sigma$$

$$= \tan^{-1}(\sigma) \Big|_s^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \boxed{\frac{\pi}{2} - \tan^{-1}(s)}$$



$$\text{if } \mathcal{L}\{f(t)\} = F(s) \quad \mathcal{L}\{g(t)\} = G(s)$$

$$\text{then } \mathcal{L}^{-1}\{F(s)G(s)\} = ?$$

work out using definition again, we get

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t f(\tau)g(t-\tau)d\tau \\ &= \int_0^t f(t-\tau)g(\tau)d\tau \\ &= f(t) * g(t)\end{aligned}$$

τ : "tau"
dummy
variables

this is the Convolution Integral of $f(t)$ and $g(t)$

example $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\}$ one Option: $\mathcal{L}^{-1} \left\{ \frac{A}{s} + \frac{Bs+C}{s^2+1} \right\}$

or

$$= \mathcal{L}^{-1} \left\{ \underbrace{\frac{1}{s}}_F \cdot \underbrace{\frac{1}{s^2+1}}_G \right\}$$

$$F = \frac{1}{s} \quad f(t) = 1$$

$$G = \frac{1}{s^2+1} \quad g(t) = \cos t \sin t$$

$$= \int_0^t f(t-\tau) g(\tau) d\tau$$

$$= \int_0^t 1 \cdot \cos \tau \sin \tau d\tau = -\cos(\tau) \Big|_0^t = 1 - \cos t$$