

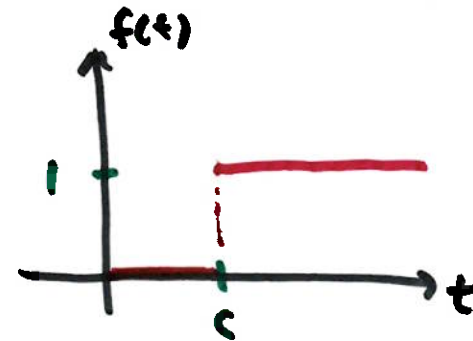
7.5 Piecewise Defined Continuous Input Functions

mass-spring-damper: $m x'' + c x' + k x = f(t)$

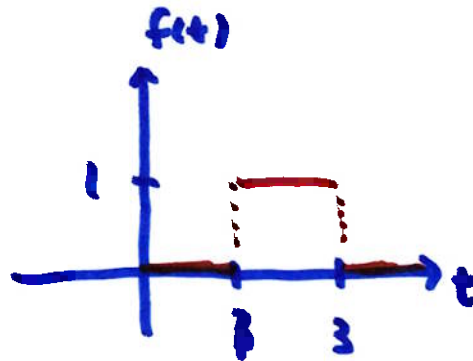
← input
mass ← damping ← spring

we use a unit step function to model discontinuous $f(t)$

$$u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{otherwise} \end{cases}$$



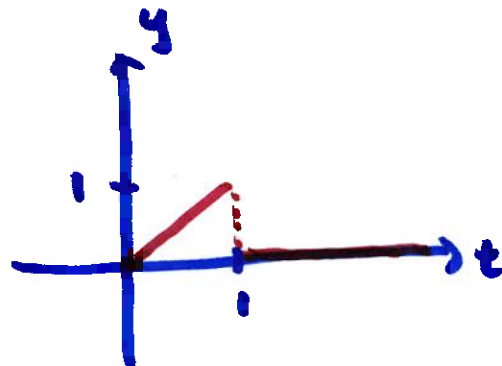
for example,



$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

$$= u(t-1) - u(t-3)$$

how about



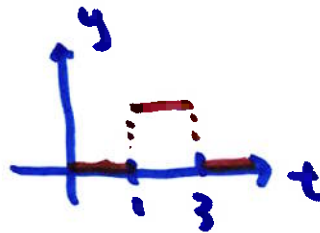
$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & t \geq 1 \end{cases} = t + u(t-1)(-t)$$

$$\mathcal{L}\{u(t-c)\} = \int_0^{\infty} u(t-c) e^{-st} dt \quad \begin{array}{l} \text{integrate } 0 \text{ up to } t=c \\ \text{integrate } 1 \text{ from } t=c \rightarrow \infty \end{array}$$

$$= \int_c^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_c^{\infty} = \frac{1}{s} e^{-cs}, \quad s > 0$$

$$\boxed{\mathcal{L}\{u(t-c)\} = e^{-cs} \cdot \frac{1}{s}}$$

hat function from earlier:



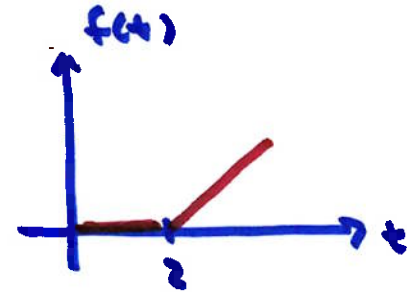
$$f(t) = u(t-1) - u(t-3)$$

$$F(s) = \mathcal{L}\{u(t-1) - u(t-3)\} = \boxed{e^{-s} \cdot \frac{1}{s} - e^{-3s} \cdot \frac{1}{s}}$$

what about $\mathcal{L}\{u(t-c)g(t)\}$? turns out it's NOT
 $e^{-cs}\mathcal{L}\{g(t)\}$

for example,

$$f(t) = \begin{cases} 0 & 0 \leq t < 2 \\ t-2 & t \geq 2 \end{cases} \\ = u(t-2)(t-2)$$



$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

$$= \int_2^{\infty} (t-2) e^{-st} dt \quad \text{by parts} = \dots = e^{-2s} \cdot \boxed{\frac{1}{s^2}}$$

is NOT $\mathcal{L}\{t-2\}$

it is actually is

$$\mathcal{L}\{t\}$$

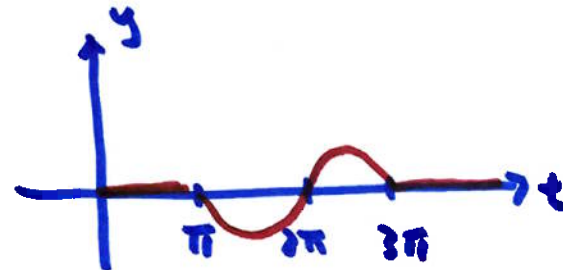
what $t-2$ actually
behaves like if it were
on at $t=0$

$$\text{so, } \boxed{\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}}$$

what results if we shift the function activated LEFT in t back to origin

example

$$f(t) = \begin{cases} 0 & t \leq \pi \\ \sin t & \pi \leq t \leq 3\pi \\ 0 & t \geq 3\pi \end{cases}$$



$$= u(t-\pi)\sin t + u(t-3\pi)(-\sin t)$$

$$= u(t-\pi)\sin t - u(t-3\pi)\sin t$$

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} - e^{-3\pi s}\mathcal{L}\{\sin(t+3\pi)\}$$

shift LEFT by π means
change t to $t+\pi$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$= e^{-\pi s}\mathcal{L}\{-\sin t\} - e^{-3\pi s}\mathcal{L}\{-\sin t\}$$

$$= e^{-\pi s} \cdot -\frac{1}{s^2+1} - e^{-3\pi s} \cdot -\frac{1}{s^2+1}$$

$$= (e^{-3\pi s} - e^{-\pi s}) \left(\frac{1}{s^2+1} \right)$$

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$$

transform after changing t to $t+c$

inverse transform: do the above in reverse

inverse transform, then shift forward in time
(change t to $t-c$)

example $F(s) = e^{-3s} \left(\frac{1}{s} + \frac{1}{s+4} \right)$ $\rightarrow$ inv. LT $\Rightarrow 1 + e^{-4t}$
then change t to $t-3$

$$f(t) = u(t-3) (1 + e^{-4(t-3)})$$

Example

$$F(s) = \underbrace{\frac{1}{s^2}}_t - 3e^{-2s} \underbrace{\frac{1}{s^2}}_t + 2e^{-3s} \underbrace{\frac{1}{s^2}}_t$$

$$f(t) = t - 3u(t-2) \cdot (t-2) + 2u(t-3) \cdot (t-3)$$

Laplace: e^{-cs} for $u(t-c)$, transform function AFTER
changing t to $t+c$

Inv. Laplace: $e^{-cs} \rightarrow u(t-c)$, inv. LT then change t to $t-c$