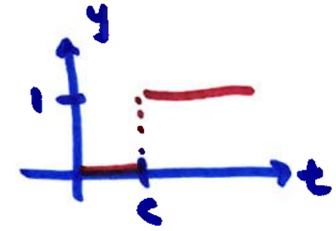
7.6 Impulse Functions

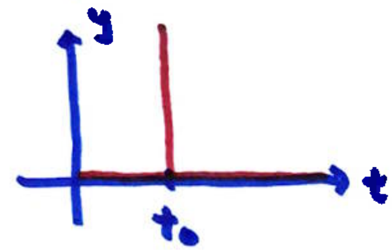
step function: $u(t-c)$



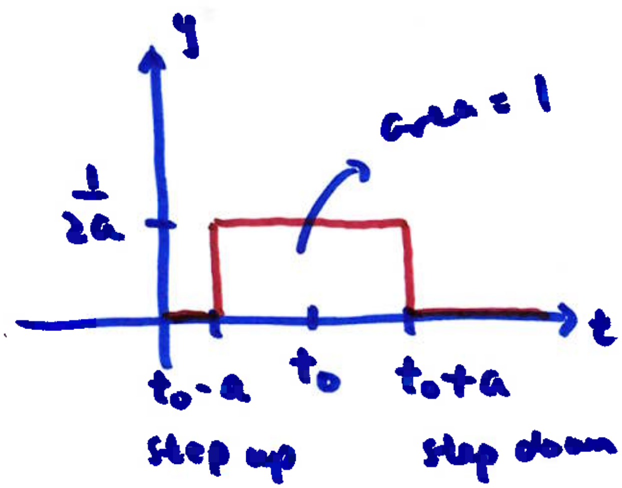
stays on once on

impulse function: very short acting input

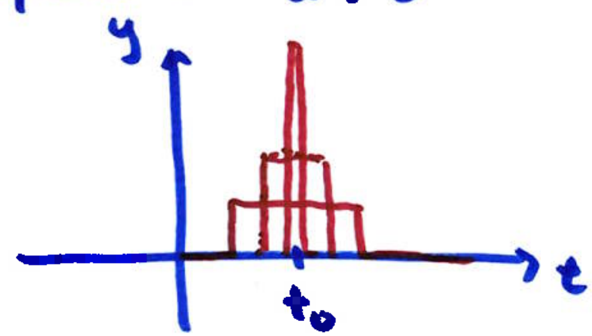
$\delta(t-t_0)$



we can construct impulse from step functions



→ impulse as $a \to 0$



impulse: $\delta(t-t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

$\mathcal{L}\{\delta(t-t_0)\} = ?$

$$= \mathcal{L}\left\{ \lim_{a\to 0}\left(\frac{1}{2a} u(t-(t_0-a)) - \frac{1}{2a} u(t-(t_0+a))\right)\right\}$$

$$= \frac{1}{2a}\left(\lim_{a\to 0}\left(\int_{t_0-a}^{\infty} e^{-st}\,dt - \int_{t_0+a}^{\infty} e^{-st}\,dt\right)\right)$$

$$= \frac{1}{2a}\lim_{a\to 0}\left(-\frac{1}{s}e^{-st}\Big|_{t_0-a}^{\infty} + \frac{1}{s}e^{-st}\Big|_{t_0+a}^{\infty}\right)$$

$$= \lim_{a\to 0}\frac{1}{2a}\left(\frac{1}{s}e^{-(t_0-a)s} - \frac{1}{s}e^{-(t_0+a)s}\right)$$

$\vdots$

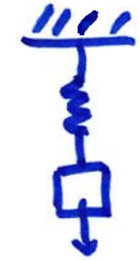
$$= e^{-t_0 s}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

$\mathcal{L}\{u(t-t_0)\} = \frac{e^{-t_0 s}}{s}$

note if $t_0 = 0$ $\mathcal{L}\{\delta(t)\} = 1$

<u>example</u> $x'' + x = 1 + \delta(t-\pi)$ $\quad x(0) = x'(0) = 0$



gravity
acc = 1
then impulse
at $t=\pi$

take Laplace of both sides

$$\left[s^2 X - s\underbrace{\cancel{x(0)}}_{x(0)=0} - \underbrace{\cancel{x'(0)}}_{x'(0)=0}\right] + X = \frac{1}{s} + e^{-\pi s}$$

$$(s^2+1) X = \frac{1}{s} + e^{-\pi s}$$

$$X = \underbrace{\frac{1}{s(s^2+1)}}_{\frac{1}{s} - \frac{s}{s^2+1}} + \underbrace{e^{-\pi s} \cdot \frac{1}{s^2+1}}$$

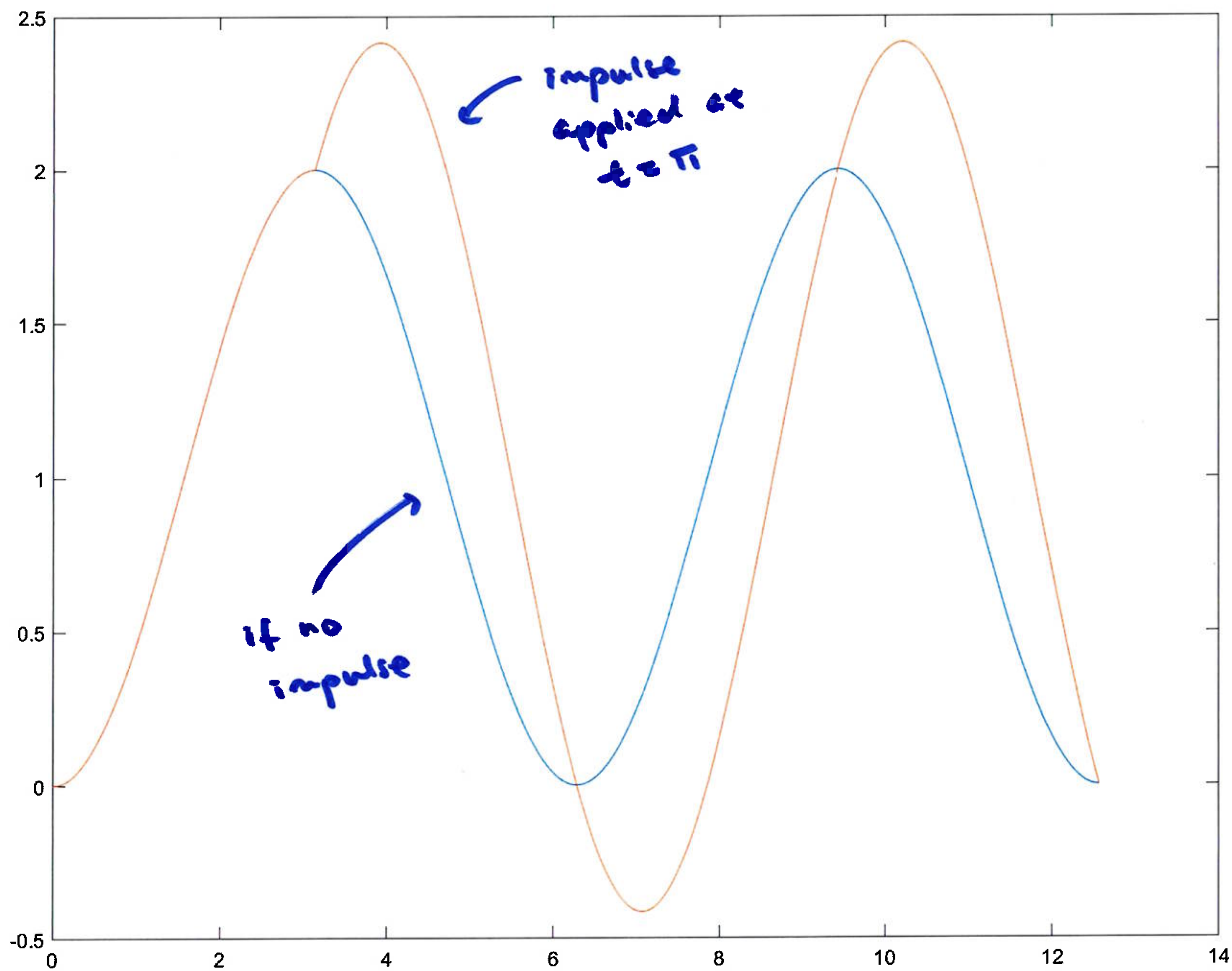
inverse LT : first find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$\to u(t-\pi)$ then $\sin(t) \to \sin(t-\pi)$

note in t domain
no impulse but step

$$x(t) = 1 - \cos t + u(t-\pi)\underbrace{\sin(t-\pi)}_{-\sin t}$$

$$= \boxed{1 - \cos t - u(t-\pi)\sin t} = \begin{cases} 1 - \cos t & 0 \le t < \pi \\ 1 - \cos t - \sin t & t \ge \pi \end{cases}$$



2.5
2
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
12
14
impulse applied at t = π
if no impulse

__example__ $x'' + 8x' + 17x = \delta(t-\pi) + \delta(t-3\pi) \quad x(0)=0,\ x'(0)=3$

$$\left[s^2 X - \cancel{s x(0)} - x'(0)\right] + 8\left[sX - \cancel{x(0)}\right] + 17X = e^{-\pi s} + e^{-3\pi s}$$

$$(s^2+8s+17)X = 3 + e^{-\pi s} + e^{-3\pi s}$$

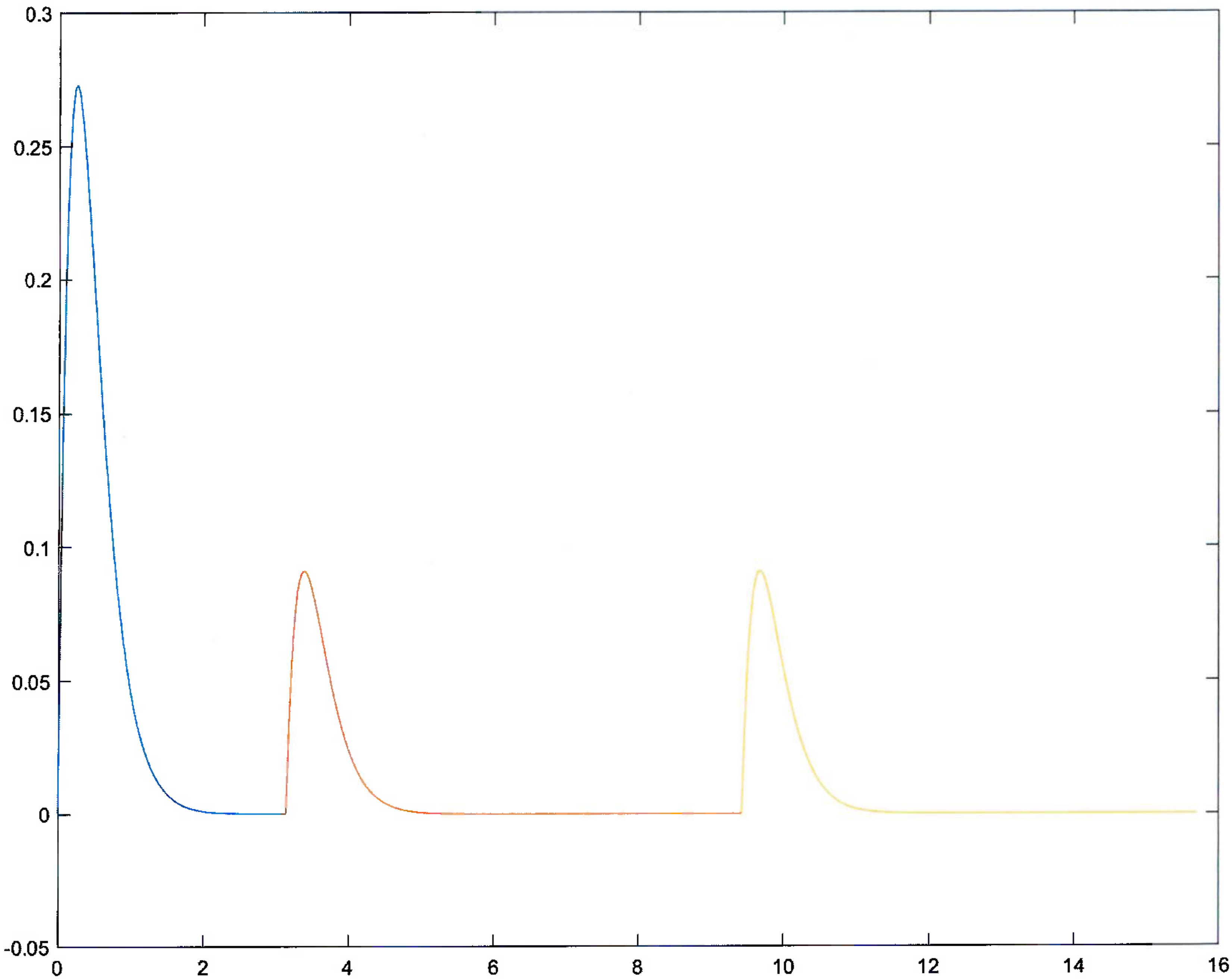
$$X = \frac{3}{s^2+8s+17} + e^{-\pi s}\frac{1}{s^2+8s+17} + e^{-3\pi s}\frac{1}{s^2+8s+17}$$

$$\rightarrow \frac{1}{(s+4)^2+1}$$

$$x(t) = 3e^{-4t}\sin t + u(t-\pi)e^{-4(t-\pi)}\sin(t-\pi) + u(t-3\pi)e^{-4(t-3\pi)}\sin(t-3\pi)$$

$$= \boxed{3e^{-4t}\sin t - u(t-\pi)e^{4\pi}e^{-4t}\sin t - u(t-3\pi)e^{12\pi}e^{-4t}\sin(t)}$$

$$= \begin{cases} 3e^{-4t}\sin t & 0 \le t < \pi \\ 3e^{-4t}\sin t - e^{4\pi}e^{-4t}\sin t & \pi \le t < 3\pi \\ \text{all} & t \ge 3\pi \end{cases}$$


0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
2
4
6
8
10
12
14
16

$$y'' + ay' + by = f(t)$$

assume $\mathcal{L}\{f(t)\} = F(s)$ exists

$y(0) = y'(0) = 0$

$$\Big\downarrow \text{LT}$$

$$s^2Y + asY + bY = F$$

$$(s^2 + as + b)Y = F$$

y : output of system

f : input

output / input $\quad \dfrac{Y}{F} = \dfrac{1}{s^2+as+b}$

from system parameters

"Transfer function"

→ impulse response of the system

solve w/o knowing $f(t)$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s^2+as+b}}_{\mathcal{L}\{h(t)\}} \cdot \underbrace{F}_{\mathcal{L}\{f(t)\}}\Big\} = \int_0^t h(t-\tau)f(\tau)\,d\tau = \int_0^t h(\tau)f(t-\tau)\,d\tau = y$$

this is looking at $f(t)$ as a bunch of impulses

→ "Duhamel's principle"