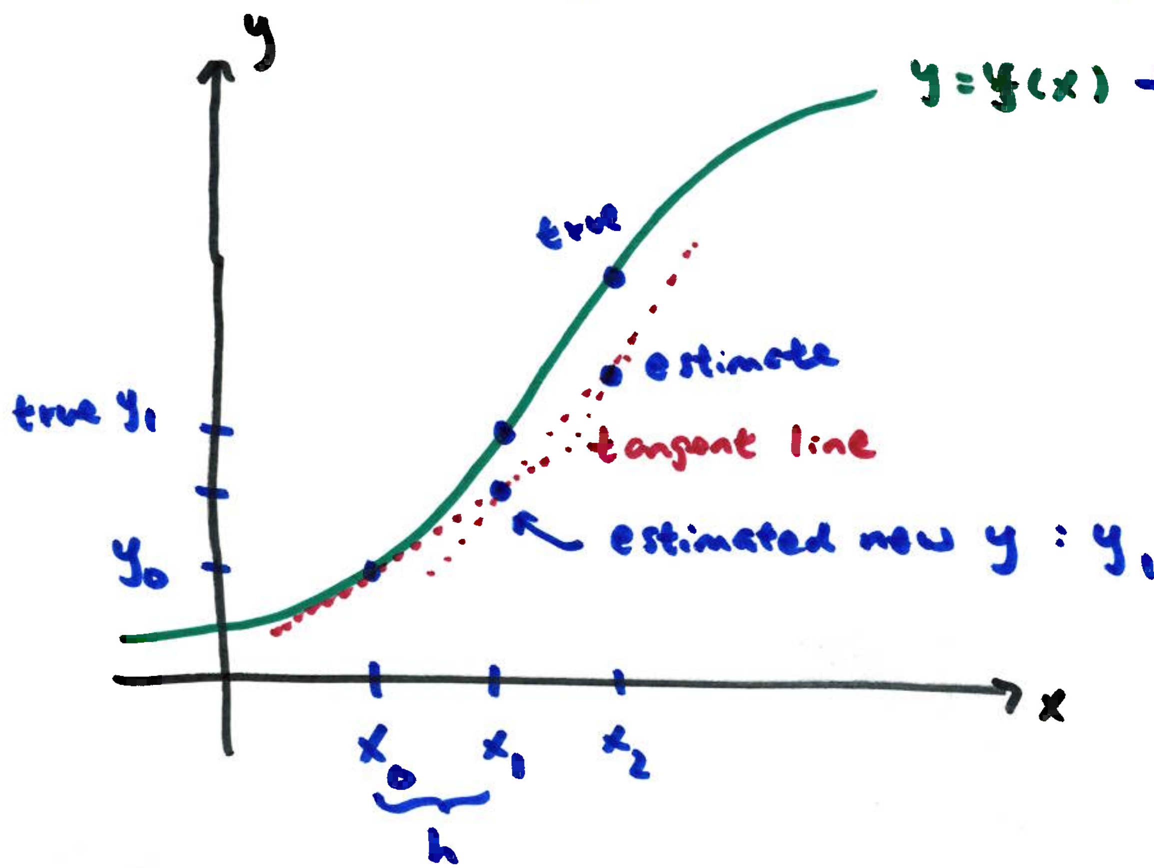
2.4 Euler's Method

one <u>numerical methods</u> to solve $y' = f(x,y)$

↳ use when impossible or impractical to solve analytically

e.g. $y' = \dfrac{x + 2y^2 - 10}{\tan xy}$

Euler's method → tangent line method (basically tangent line approx.)



$y = y(x)$ → what we want but we don't have

(x_0, y_0) : initial condition

travel to ~~y0~~ $x_1 = x_0 + h$ ← step size

estimated y_1, then travel on new tangent line

repeat until reach target x

tangent line at (x_0, y_0) $\qquad y - y_0 = f'(x_0, y_0)(x - x_0)$

$$\boxed{y = y_0 + f'(x_0, y_0)(h)}$$ Euler update equation

Algorithm

$x_0 = \text{given}$
$y_0 = \text{given}$ } initial condition

$x_1 = x_0 + h$ $\qquad$ h: decided, fixed

$y_1 = y_0 + f'(x_0, y_0)\, h$

$x_2 = x_1 + h$

$y_2 = y_1 + f'(x_0, y_1)\, h$

$\vdots$

$y_{n+1} = y_n + f'(x_n, y_n)\, h$ $\qquad$ stop at target x_n

example

$y' = 2y - 3x \qquad y(0) = 1$

$x_0 \quad y_0 \quad$ target x

use $h = 0.25$ to estimate $y(0.5)$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + f'(x_0, y_0)\,h$

$$= 1 + [\underbrace{2(1) - 3(0)}_{\text{use old } x,\, y}](0.25) = 1.5$$

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$ target x reached

$y_2 = y_1 + f'(x_1, y_1)\,h$

$$= 1.5 + [2(1.5) - 3(0.25)](0.25) = \boxed{2.0625}$$

how good/bad ?

we can solve $y' = 2y - 3x \quad y(0) = 1$ analytically (which is usually not the case)

$y' - 2y = -3x$ $\qquad$ $y' + p(x)y = g(x)$

integrating factor: $I = e^{\int p\,dx} = e^{\int -2\,dx} = e^{-2x}$

$\frac{d}{dx}(Iy) = -3xI$

$Iy = \int -3x\,I\,dx$

$e^{-2x}y = \int -3xe^{-2x}\,dx$

$\vdots$

$y = \frac{3}{4}(2x+1) + \frac{1}{4}e^{2x}$

true $y(0.5) = 2.1796$ $\qquad$ est. using $h = 0.25$

$y = 2.0625$

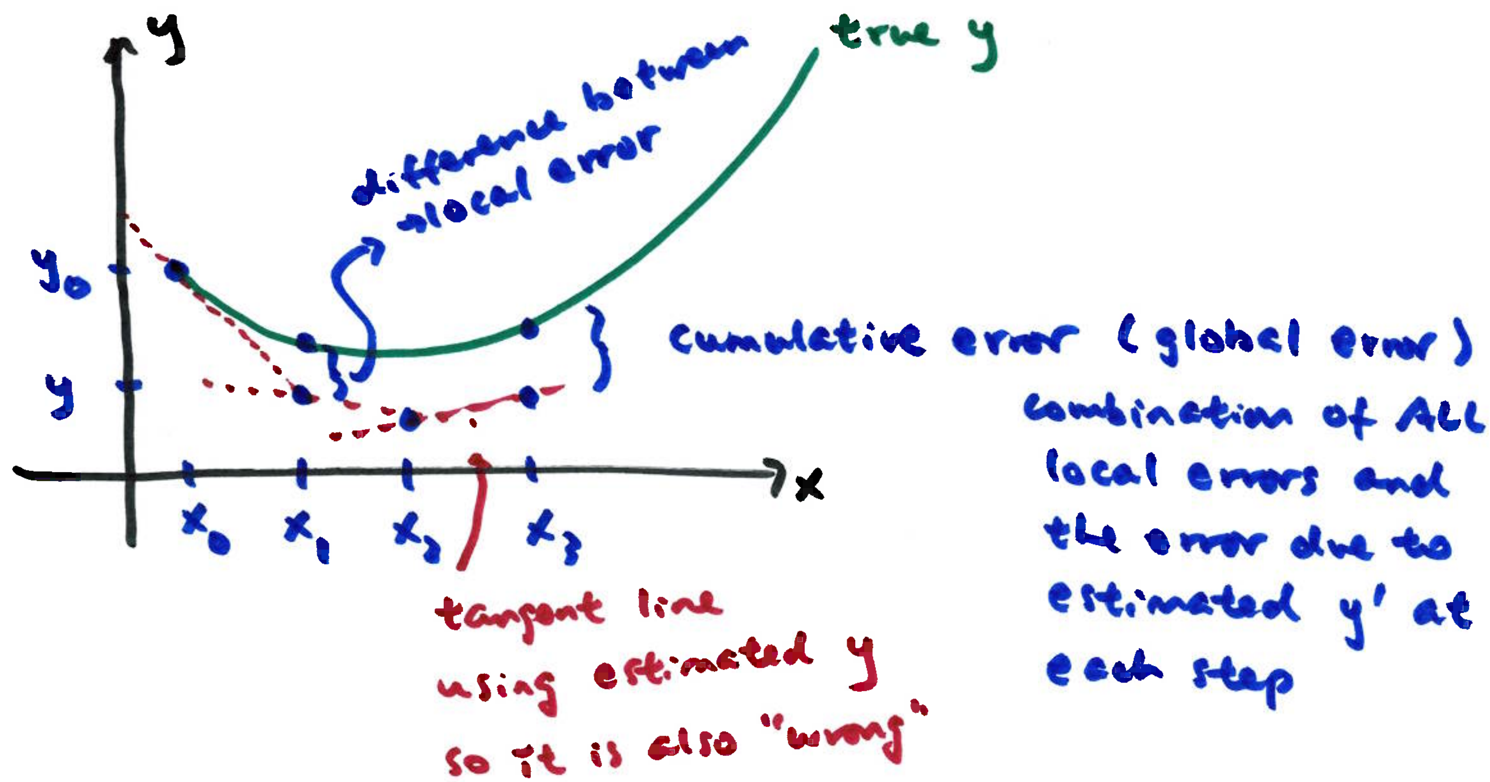
improve estimate?

$\rightarrow$ shrink h

if $h = 0.05$, 10 steps later, $y(0.5) \approx 2.1484$

if $h = 0.01$ $\quad$ $y(0.5) \approx 2.1729$

Sources of error



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in general, we don't know true y

how do we know the estimation is "good enough"?

→ shrink h until the estimated y doesn't change significantly ("solution has converged")

$y' = 2y - 3x \quad y(0) = 1$, estimate $y(1)$

$h = 0.5 \quad y(1) \approx 3.25$

$h = 0.05 \quad y(1) \approx 3.932$

} significant, keep shrinking h

$h = 0.01 \quad y(1) = 4.061$

$h = 0.001 \quad y(1) = 4.094$

} settling down?

$h = 0.0005 \quad y(1) = 4.095$

} practically not changing
stop.

Wolfram syntax:

use Euler method $y' =$ ____, $y(\_) =$ _, from _ to _, $h =$ _

e.g.

use Euler method $y' = y$, $y(0) = 1$, from 0 to 2, $h = 0.01$