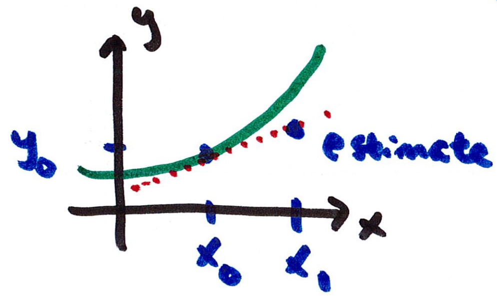
2.5 Improved Euler Method

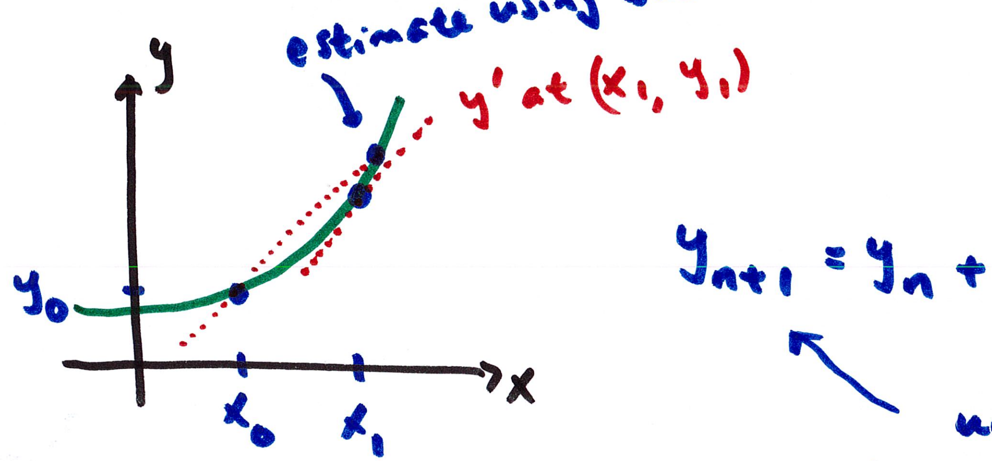
last time: Euler's method $y_{n+1} = y_n + f(x_n, y_n)h$



problem: errors accumulate as we move and can cause instability if h is small

root cause of error: estimation of slope

Backward Euler Method



$$y_{n+1} = y_n + f(x_{n+1}, y_{n+1})h$$

unknown

how to evaluate y_{n+1}?

need to solve

Forward Euler is an <u>explicit</u> method

Backward Euler is an <u>implicit</u> method → need to use y_{n+1} which we don't know

→ generally requires root finding process

<u>example</u> $\quad y' = 2y - 3x \qquad y(0) = 1, \quad h = 0.1$

Backward Euler: $y_{n+1} = y_n + f(x_{n+1}, y_{n+1})h$

$$x_0 = 0$$

$$y_0 = 1$$

$$x_1 = 0 + 0.1 = 0.1$$

$$y_{n+1} = y_n + f(x_{n+1}, y_{n+1})h$$

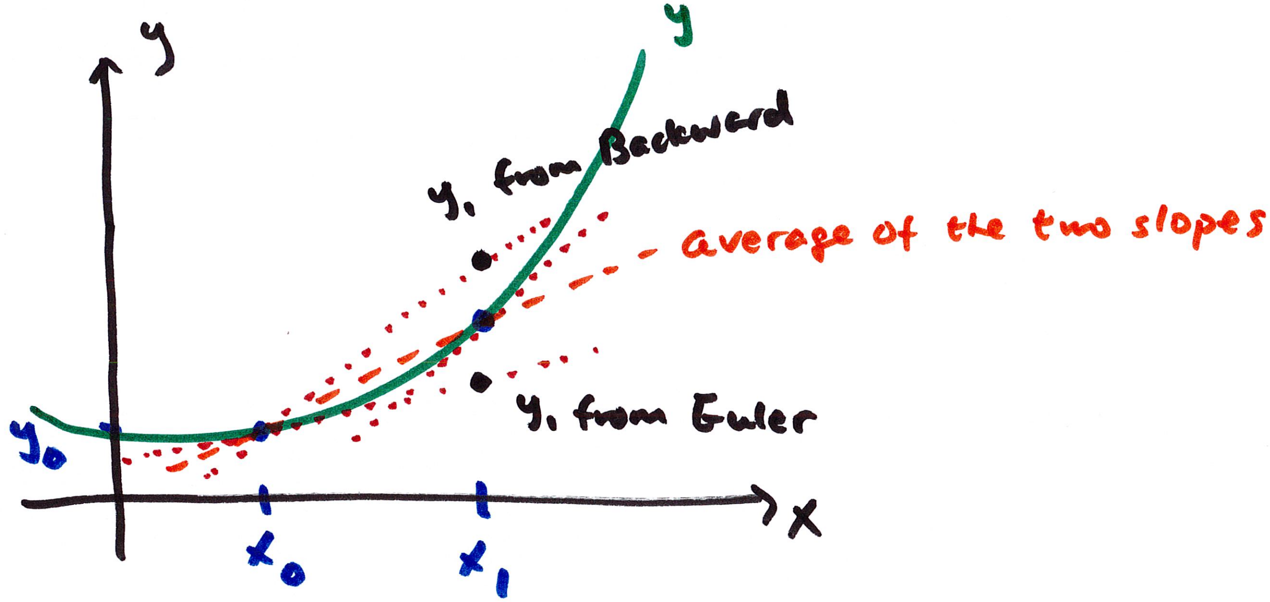
$$y_1 = \underset{\uparrow\, 1}{y_0} + [2y_1 - 3(0.1)](0.1)$$

$$= 1 + 0.2y_1 - 0.03$$

$$0.8y_1 = 0.97 \rightarrow y_1 = 1.2125$$

Improved Euler (or Heun's) Method

basic idea: use the average of $f(x_n, y_n)$ and $f(x_{n+1}, y_{n+1})$ to move forward



update formula:

$$\boxed{y_{n+1} = y_n + \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] h}$$

again, we don't have y_{n+1} which we need on the right

so, we use the Euler approx. for that

$$y_{n+1} = y_n + \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + f(x_n, y_n)h)\right] h$$

<u>Algorithm</u>

x_0 = given

y_0 = given

decide h

predict y_{n+1} using Euler : $y_{n+1} = y_n + f(x_n, y_n) h$

correct y_{n+1} using Improved Euler

$$y_{n+1} = y_n + \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] h$$

then $x_1 = x_0 + h$

etc

this is an example of the Predictor - Corrector Method

<u>example</u> $y' = 2y - 3x \quad y(0) = 1$

use $h = 0.05$ to estimate $y(0.1)$

$x_0 = 0$

$y_0 = 1$

$x_1 = 0 + 0.05 = 0.05$

predict y_1: $y_1 = y_0 + [2(y_0) - 3(x_0)]h$

$= 1 + [2(1) - 3(0)](0.05) = 1.1$

correct y_1: $y_1 = y_0 + \frac{1}{2}[(2(y_0) - 3(x_0)) + (2(y_1) - 3(x_1))]h$

$= 1 + \frac{1}{2}[(2 \cdot 1 - 3 \cdot 0) + (2 \cdot 1.1 - 3 \cdot 0.05)](0.05)$

$= 1.10125$

$x_2 = 0.1$ (target)

predict y_2: $y_2 = 1.10125 + (2 \cdot 1.10125 - 3 \cdot 0.05)(0.05)$

$= 1.203875$

correct y_2: $y_2 = 1.10125 + \frac{1}{2}[(2 \cdot 1.10125 - 3 \cdot 0.05)$

$+ (2 \cdot 1.203875 - 3 \cdot 0.1)](0.05)$

$y_2 = 1.20525625$ compare to Euler : 1.2025

true : 1.20535

MA 303 Exam 1

- 8 pm on Tuesday, 2/20 in MTHW 210
- Covers up to section 2.4 (Euler's Method)
- In the neighborhood of 12 questions
  - "Hybrid multiple-choice"
    - No guessing, work required
    - Wrong answer but "good" work—50% partial credit
- One 4 in x 6 in notecard allowed
  - Must be handwritten
  - No sharing
  - Each of the six surfaces must be used two-dimensionally
- Table of Laplace transforms will be provided