9.2 Fourier Series (part 2)

$f(t)$ periodic with period of 2π on $-\pi \le t \le \pi$

can be written as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nt) + b_n\sin(nt)\right]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

example from last time

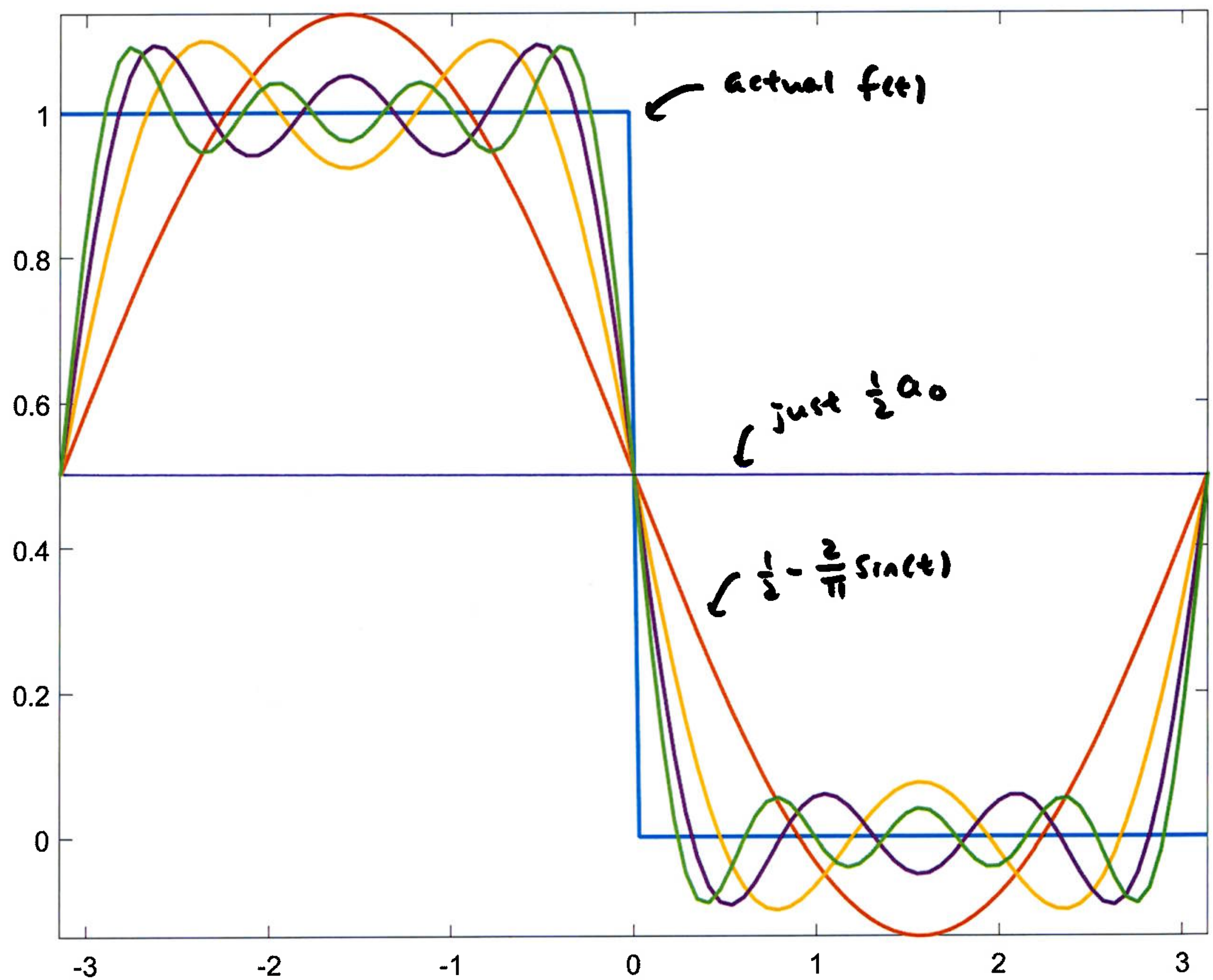
$$f(t) = \begin{cases} 1 & \text{if } -\pi \le t \le 0 \\ 0 & \text{if } 0 \le t < \pi \end{cases} \quad \text{period of } 2\pi$$

we calculated $a_0 = 1$, $a_n = 0$, $b_n = \frac{-1+(-1)^n}{n\pi} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-2}{n\pi} & \text{if } n \text{ odd} \end{cases}$

Fourier series is

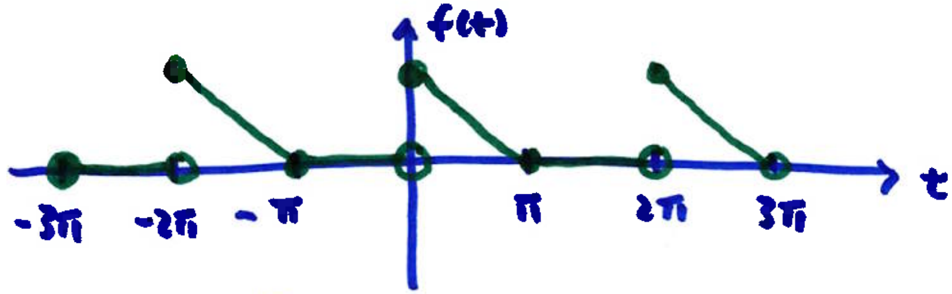
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nt) + b_n\sin(nt)\right] \rightarrow \text{converges to actual } f(t) \text{ if } n\to\infty$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-1+(-1)^n}{n\pi}\sin(nt) = \frac{1}{2} - \frac{2}{\pi}\sin(t) - \frac{2}{3\pi}\sin(3t) - \frac{2}{5\pi}\sin(5t) + \dots$$



actual f(t)
just ½ a₀
½ - 2/π sin(t)
-3
-2
-1
0
1
2
3
0
0.2
0.4
0.6
0.8
1

<u>example</u> $f(t) = \begin{cases} 0 & -\pi \le t < 0 \\ \pi - t & 0 \le t < \pi \end{cases}$ period 2π



$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$

calculate a_0 first $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} (\pi - t)\,dt = \frac{1}{2}\pi$

$a_n = \frac{1}{\pi}\int_0^{\pi} (\pi - t)\cos(nt)\,dt$ by parts $\quad u = \pi - t \quad dv = \cos(nt)\,dt$

$du = -dt \quad v = \frac{1}{n}\sin(nt)$

$= \frac{1}{\pi}\left((\pi - t)\frac{1}{n}\sin(nt)\Big|_0^{\pi} + \int_0^{\pi} \frac{1}{n}\sin(nt)\,dt \right)$

$= \frac{1}{\pi}\left(-\frac{1}{n^2}\cos(nt)\Big|_0^{\pi} = -\frac{1}{n^2\pi}\Big(\underbrace{\cos(n\pi)}_{(-1)^n} - 1\Big) = \frac{1}{n^2\pi}\left(1 - (-1)^n\right)\right.$

if n even $a_n = 0$

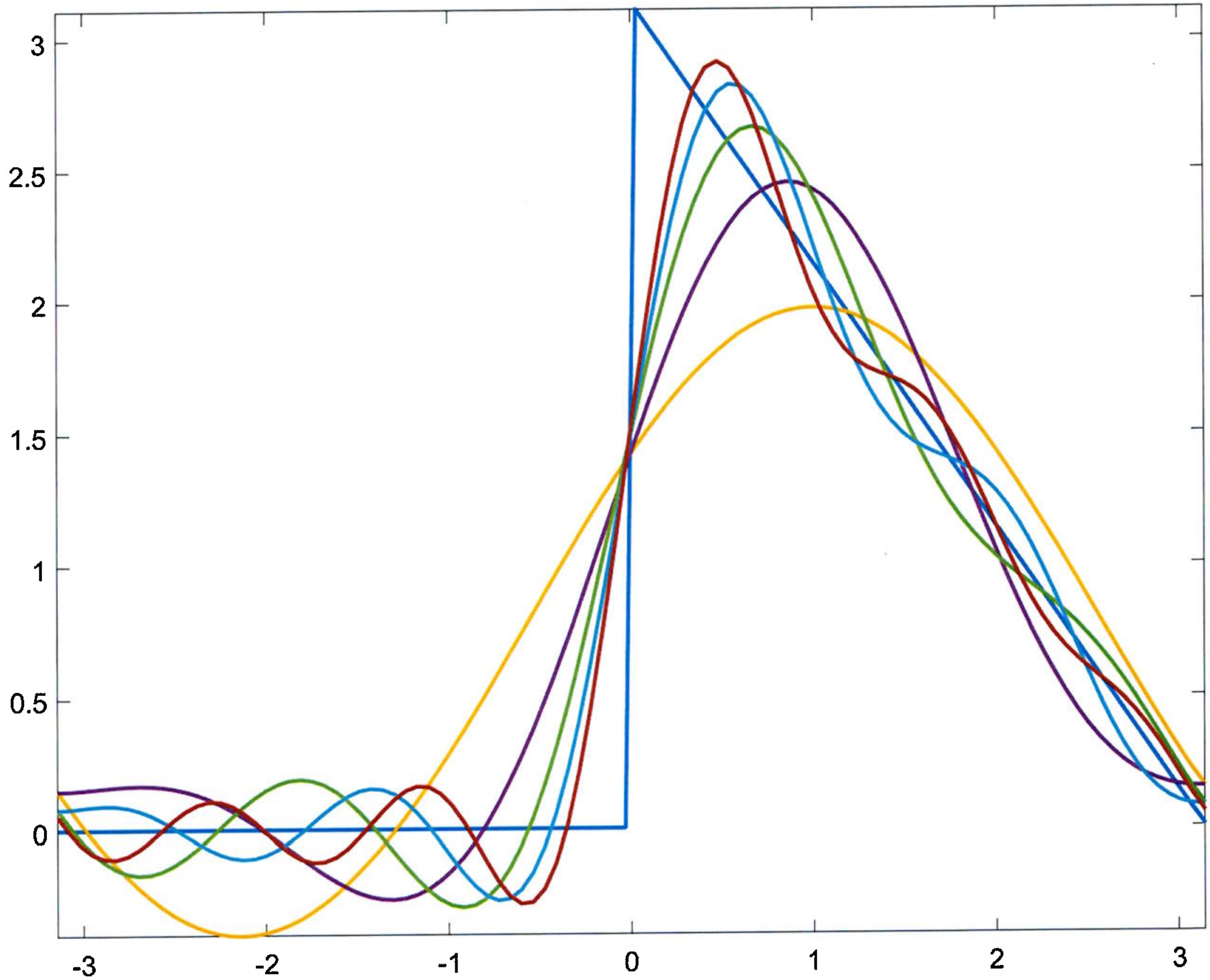
n odd $a_n = \frac{2}{n^2\pi}$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - t)\sin(nt)\,dt = \cdots = \frac{1}{n}$$

$$f(t) = \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1-(-1)^n)\cos(nt) + \frac{1}{n}\sin(nt)\right]$$

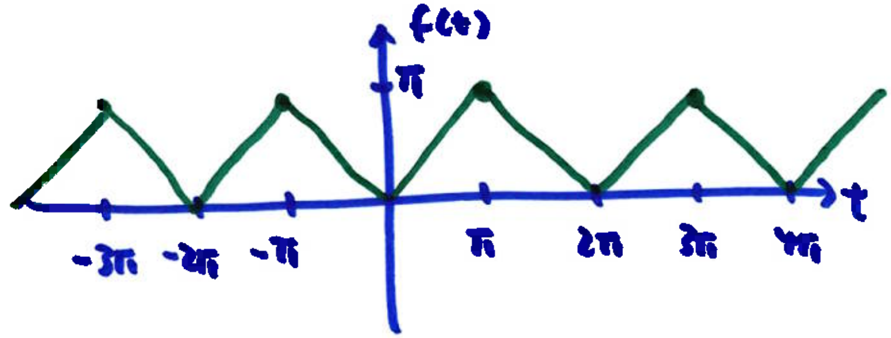
$$= \frac{1}{4}\pi + \frac{2}{\pi}\cos(t) + \sin(t) + \frac{1}{2}\sin(2t) + \frac{2}{9\pi}\cos(3t) + \frac{1}{3}\sin(3t)$$

$$+ \frac{1}{4}\sin(4t) + \frac{2}{25\pi}\cos(5t) + \frac{1}{5}\sin(5t) + \cdots$$


3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2
3

<u>example</u> $f(t) = |t|$ $\quad -\pi \le t \le \pi \quad$ period 2π

$$f(t) = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \ge 0 \end{cases}$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

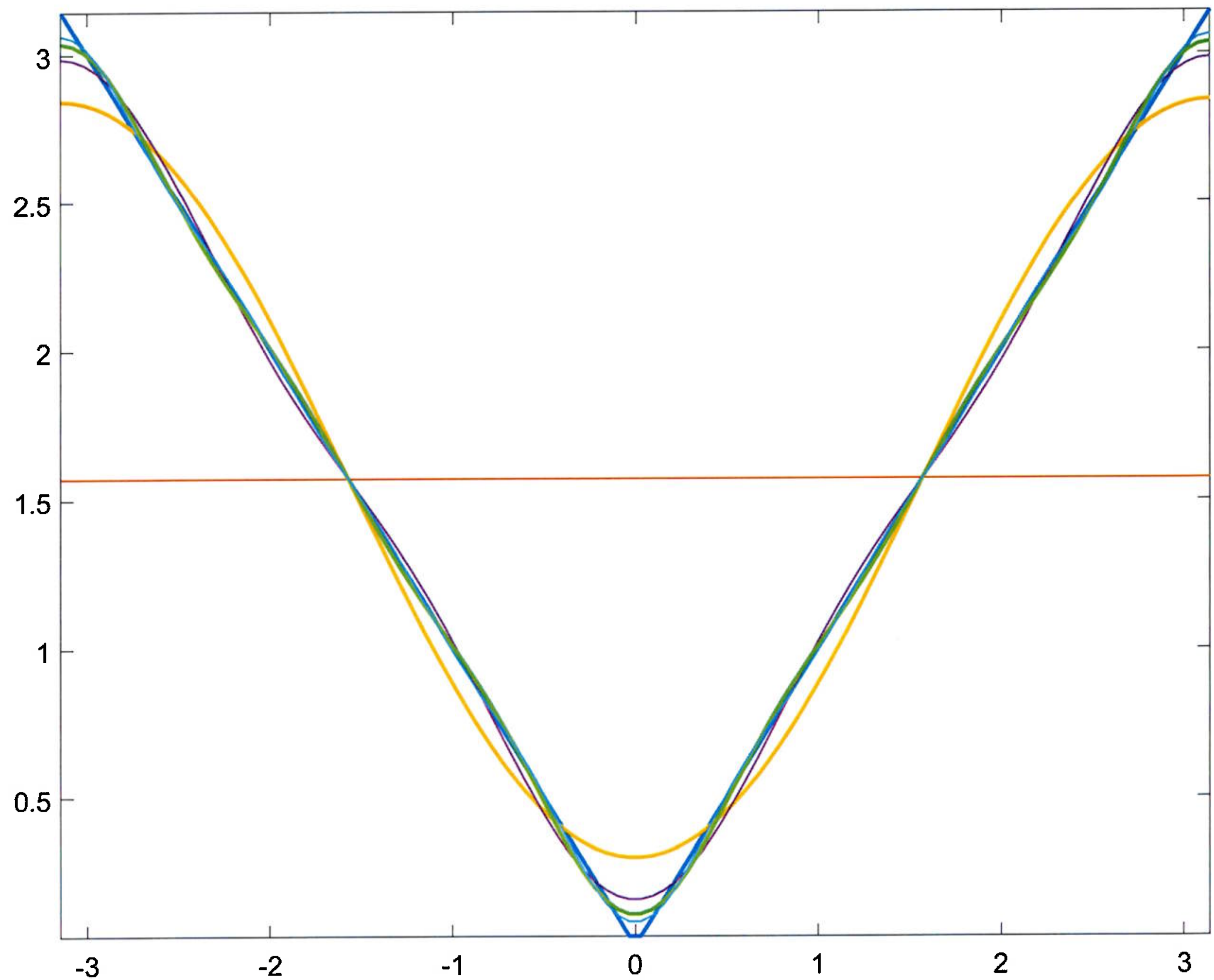
$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -t\,dt + \int_{0}^{\pi} t\,dt\right) = \dots = \pi$$

$$a_n = \dots = \frac{2}{n^2\pi}\left((-1)^n - 1\right) \qquad b_n = \dots = 0$$

Fourier series is

$$\frac{1}{2}\pi + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos(nt) = \frac{1}{2}\pi - \frac{4}{\pi}\cos(t) - \frac{4}{9\pi}\cos(3t)$$

$$- \frac{4}{25\pi}\cos(5t) + \dots$$


3
2.5
2
1.5
1
0.5
-3
-2
-1
0
1
2
3