5.5 Multiple Eigenvalue Solutions

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvalues: 1, 1 (algebraic multiplicity of 2)

eigenvectors: solve $(A-\lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$A - \lambda I = A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(A-\lambda I)\vec{v} = \vec{0}$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ both free

$a = r$

$b = s$

$\vec{v} = \begin{bmatrix} r \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ choose $r, s = 1$ so

eigenevctors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

geometric multiplicity of 2

if geo. mult. = alg. mult. $\rightarrow$ A is <u>complete</u>

Solution: $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$

$$= \boxed{c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

try $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = 1, 1 \quad$ (alg. mult. = 2)

$(A - \lambda I)\vec{v} = \vec{0} \qquad \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad a = r$, $b = 0$

$$= \begin{bmatrix} r \\ 0 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

geo. mult. = 1

geo. mult. < alg. mult. $\rightarrow$ A is <u>defective</u>

Solution: $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$?

motivated by scalar case: $y'' + 10y' + 25y = 0$

characteristic eq. : $r^2 + 10r + 25 = 0$

$(r+5)(r+5) = 0$

$r = -5, -5$

solution: $y = c_1 e^{-5t} + c_2 t e^{-5t}$

turns out we can do similar thing for system $\vec{x}\,' = A\vec{x}$

$\vec{x}$ ~~r/fn~~ fundamental solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1$

$$\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

sub $\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$ into $\vec{x}\,' = A\vec{x}$

$$\vec{x}_2\,' = \lambda e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) + e^{\lambda t}(\vec{v}_1)$$

$$\lambda e^{\lambda t} t\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2 + e^{\lambda t}\vec{v}_1 = A e^{\lambda t} t\vec{v}_1 + A e^{\lambda t}\vec{v}_2$$

so, $A e^{\lambda t}\vec{v}_1 = \lambda e^{\lambda t}\vec{v}_1 \longrightarrow A\vec{v}_1 = \lambda\vec{v}_1 \rightarrow (A - \lambda I)\vec{v}_1 = \vec{0}$

$$\lambda e^{\lambda t}\vec{v}_2 + e^{\lambda t}\vec{v}_1 = A e^{\lambda t}\vec{v}_2$$

$\lambda \vec{v}_2 + \vec{v}_1 = A\vec{v}_2 \longrightarrow \boxed{(A-\lambda I)\vec{v}_2 = \vec{v}_1}$ find $\vec{v}_2$

back to $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

we found $\lambda = 1, 1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}_1 = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

now $\vec{v}_2$: $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \vec{v}_2 = \begin{bmatrix} a \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

choose $a=0$ so don't include $\vec{v}_1$

so, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_2 = e^t \Big(t \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \Big)$

general solution : $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \Big(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big)$

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$ multiply by $(A-\lambda I)$

$(A-\lambda I)(A-\lambda I)\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \vec{0}$

$(A-\lambda I)^2\vec{v}_2 = \vec{0}$

in general, $(A-\lambda I)^{k+1}\vec{v} = \vec{0}$ k: # of missing vectors

solve $(A-\lambda I)^{k+1}\vec{v} = \vec{0}$ for the "top" missing, then step "down" by solving $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ for the rest.

this is more efficient way for 3×3

<u>example</u> $\vec{x}' = \begin{bmatrix} -13 & 0 & -4 \\ -1 & -11 & -1 \\ 1 & 0 & -9 \end{bmatrix} \vec{x}$

$\lambda = -11, -11, -11$ alg. mult $= 3$

solve $(A-\lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

geo. mult. $= 1$

missing 2 : $\vec{v}_2, \vec{v}_3$

$(A-\lambda I)^{2+1} \vec{v}_3 = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

anything since $[0][\vec{v}_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

choose $\vec{v}_3$ arbitrarily ~~and~~

<u>as long as</u> $(A-\lambda I)\vec{v}_3 \neq \vec{0}$

step down: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \vec{v}_2}$$

step down: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_1}$$

(matches "real" eigenvector) may not be the case

solutions: $\vec{x}_1 = e^{-1t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{\lambda t}\vec{v}_1$

$$\vec{x}_2 = e^{-1t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

$$\vec{x}_3 = e^{-1t}\left(\frac{1}{2}t^2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= e^{\lambda t}\left(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$$

general: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

<u>example</u> $\vec{x}' = \begin{bmatrix} 6 & -9 & 0 \\ 1 & 12 & 0 \\ 1 & 3 & 9 \end{bmatrix} \vec{x}$

$\lambda = 9, 9, 9$

$(A - \lambda I)\vec{v} = \vec{0}$ $\quad \left[\begin{array}{ccc|c} -3 & -9 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right]$

$\vdots$

$\vec{v} = \begin{bmatrix} -3s \\ s \\ r \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

missing 1: $\quad (A - \lambda I)^{1+1}\vec{v}_3 = \vec{0}$

↳ always $[0]$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ choose $\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

step down: $(A - \lambda I)\vec{v}_3 = \vec{v}_2$

$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \vec{v}_2}$

Step down: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ because $\vec{v}_2$ is a lin. combo of "real" EV's

if that happens, choose $\vec{v}_1$ = either of the "real" EV

$$\boxed{\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}$$

$$\vec{x}_1 = e^{+5t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_2 = e^{+5t}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_3 = e^{+5t}\left(t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$