9.2 General Fourier Series and Convergence (part 1)

Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nt) + b_n\sin(nt)\right]$$

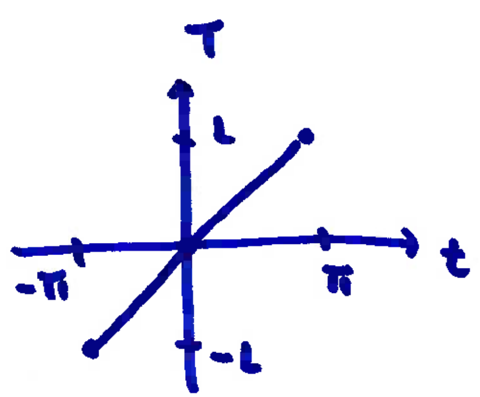
$f(t)$ period 2π on $-\pi < t < \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

now, let's relax the period 2π and $-\pi < t < \pi$ part

let some function be periodic with period $2L$ defined on $-L < t < L$

change of variable – define T such that when $t = -\pi$, $T = -L$
$t = \pi$, $T = L$



$$T = \frac{L}{\pi}t \rightarrow t = \frac{\pi}{L}T$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad t = \frac{\pi}{L}T \quad dt = \frac{\pi}{L}dT$$

$$= \frac{1}{\cancel{\pi}}\int_{-L}^{L} f(T)\cos\left(n\cdot\frac{\pi}{L}T\right)\cdot\frac{\cancel{\pi}}{L}\,dT$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(T)\cos\left(\frac{n\pi}{L}T\right)dT \qquad L:\text{ half period}$$

repeat and we get

$$b_n = \frac{1}{L}\int_{-L}^{L} f(T)\sin\left(\frac{n\pi}{L}T\right)dT$$

Fourier basis functions : $\cos(nt) \to \cos\left(\frac{n\pi}{L}T\right)$

$\sin(nt) \to \sin\left(\frac{n\pi}{L}T\right)$

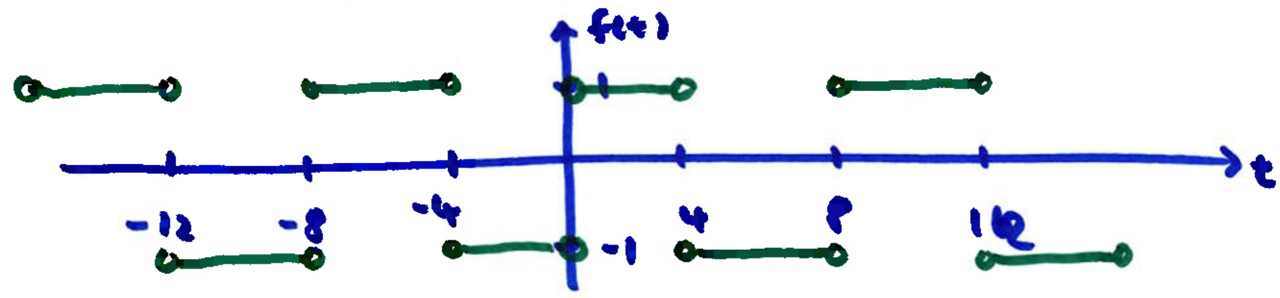
T is just a variable, let's call it t

Fourier series for periodic $f(t)$ w/ period $2L$, $-L<t<L$

is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}t\right) + b_n\sin\left(\frac{n\pi}{L}t\right)$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

<u>example</u> $f(t) = \begin{cases} -1 & -4 < t < 0 \\ 1 & 0 < t < 4 \end{cases}$ period 8



L is the half period, so $L = 4$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

start with a_0: $a_0 = \frac{1}{4}\overbrace{\int_{-4}^{4} f(t)\,dt}^{\text{area between } f(t) \text{ and } t\text{-axis}}$

$$= \frac{1}{4}\int_{-4}^{0} -1\cdot dt + \frac{1}{4}\int_{0}^{4} dt$$

$$= \frac{1}{4}(-t)\Big|_{-4}^{0} + \frac{1}{4}(t)\Big|_{0}^{4} = 0$$

$$a_n = \frac{1}{4}\left(\int_{-4}^{0} -\cos\left(\frac{n\pi}{4}t\right)dt + \int_{0}^{4} \cos\left(\frac{n\pi}{4}t\right)dt\right)$$

$$= \frac{1}{4}\left(-\frac{4}{n\pi}\sin\left(\frac{n\pi t}{4}\right)\Big|_{-4}^{0} + \frac{4}{n\pi}\sin\left(\frac{n\pi t}{4}\right)\Big|_{0}^{4}\right)$$

$$= 0$$

so no constant and no cosine terms

$$b_n = \frac{1}{4}\left(\int_{-4}^{0} -\sin\left(\frac{n\pi}{4}t\right)dt + \int_{0}^{4} \sin\left(\frac{n\pi}{4}t\right)dt\right)$$

$$= \frac{1}{4}\left(\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)\Big|_{-4}^{0} + -\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)\Big|_{0}^{4}\right)$$

$$= \frac{1}{4}\left(\frac{4}{n\pi} - \frac{4}{n\pi}\cos(n\pi) - \frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\right)$$

$$= \frac{2}{n\pi}\left(1 - \underbrace{\cos(n\pi)}_{(-1)^n}\right) = \frac{2}{n\pi}\left(1 - (-1)^n\right) \qquad n = 1, 2, 3, \ldots$$

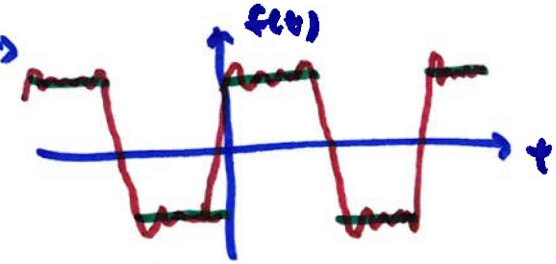
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Fourier Series: $\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin\left(\frac{n\pi}{4}t\right)$

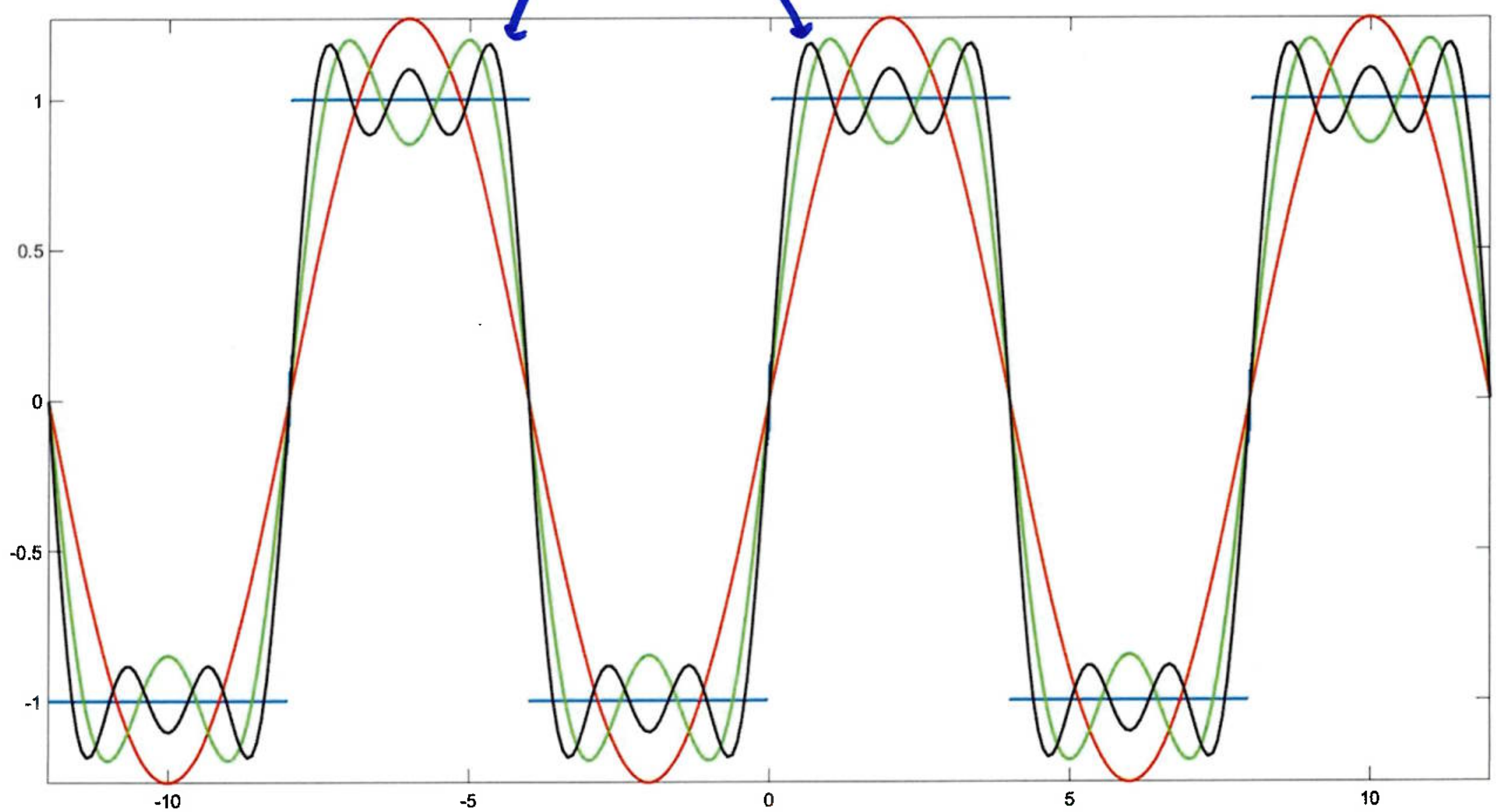
$$= \sum_{n=\text{odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right)$$

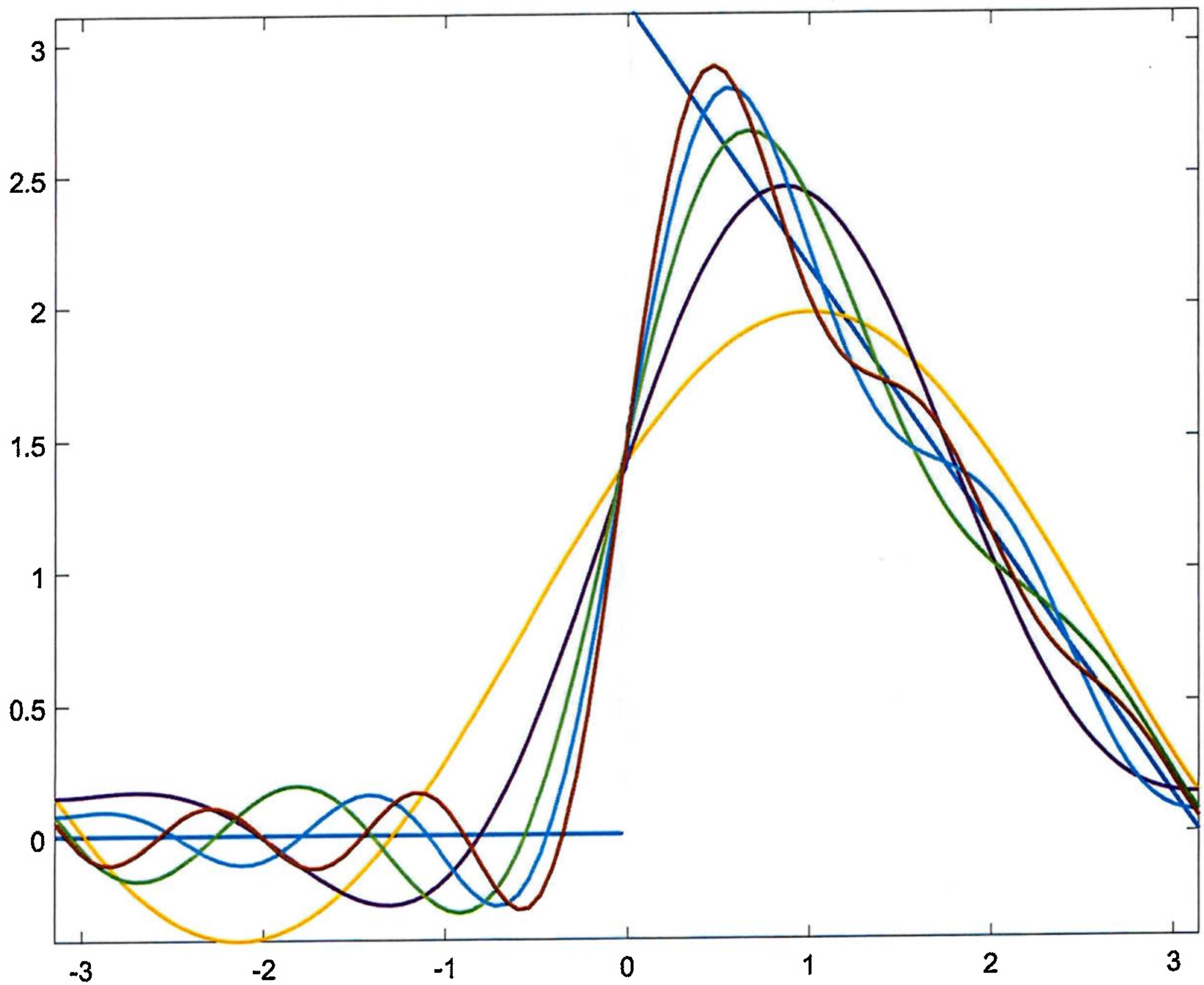
$$= \frac{4}{\pi}\sin\left(\frac{\pi t}{4}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{4}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{4}\right) + \dots$$

notice from the graph that as more terms are included, the Fourier series tracks the actual function better and better as $n \to \infty$, Fourier series converges to $f(t)$ wherever $f(t)$ is <u>continuous</u>



at discontinuities, Fourier series converges to $\frac{f(t^-)+f(t^+)}{2}$ → avg of last defined point on Left and first on Right big jumps right before and after discontinuities → Gibb's Phenomenon





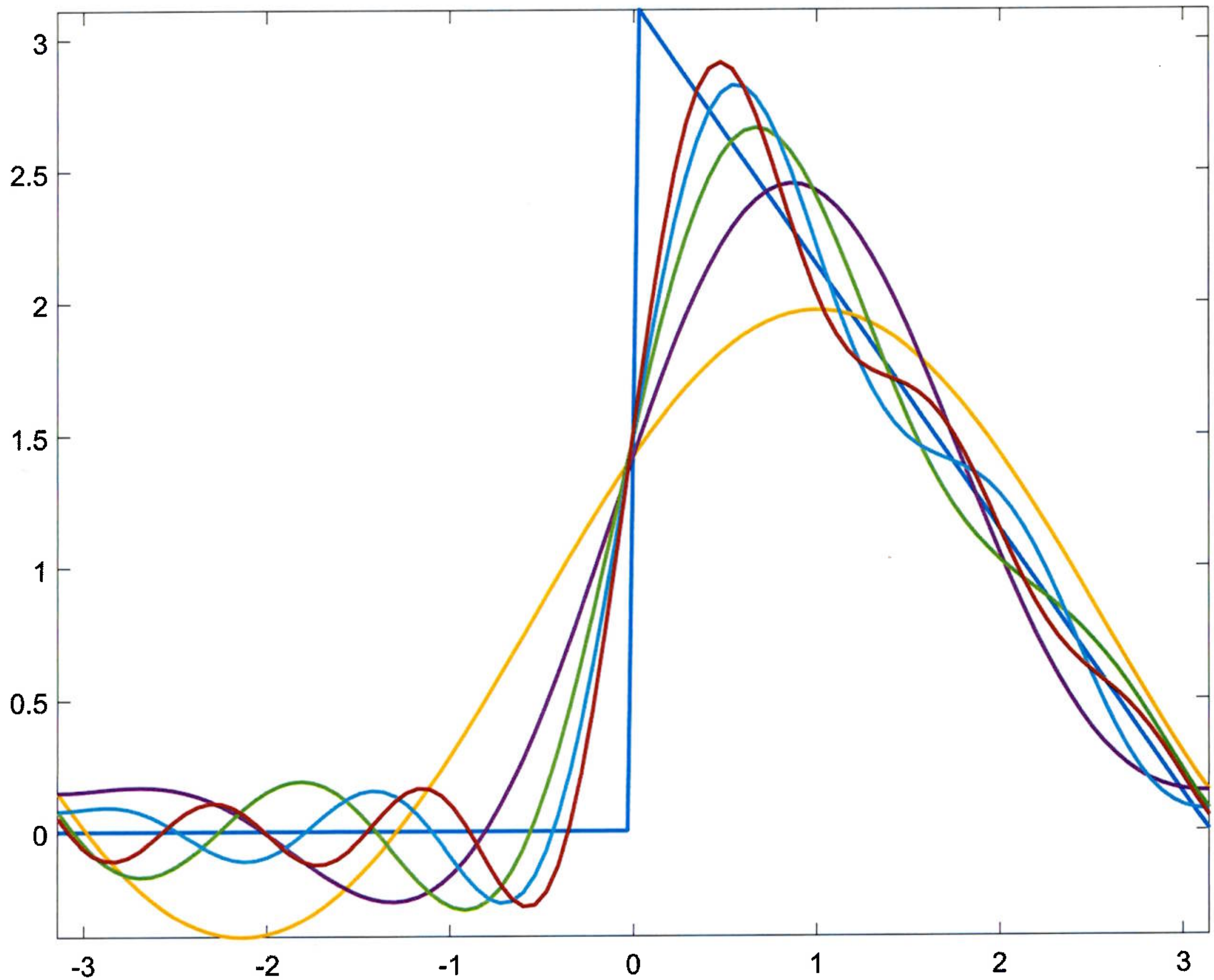
3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2
3

next extension to Fourier series formulas $\rightarrow$ remove the requirement that $-L < t < L$

in practice, we'd like to work with $f(t)$ with period $2L$ but

$0 < t < 2L$

next time $\rightarrow$ new formulas for $0 < t < 2L$


3
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