

9.3 Fourier Cosine and Sine Series (part 1)

$f(t)$ is even if $f(-t) = f(t)$ e.g. $t^2, t^4, \cos(t)$
→ y-axis symmetry

$f(t)$ is odd if $f(-t) = -f(t)$ e.g. $t, t^3, \sin(t)$
→ origin symmetry

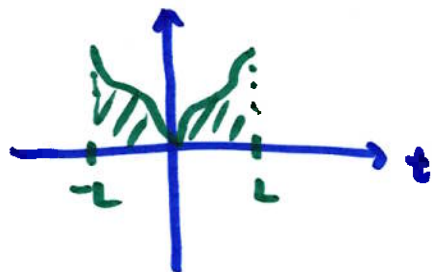
product of two even functions is even

product of two odd functions is even

product of one odd and one even is odd

$$a_n = \frac{1}{L} \int_{-L}^L \underbrace{f(t) \cos\left(\frac{n\pi t}{L}\right)}_{\text{even}} dt$$

if $f(t)$ is even, then $f(t) \cos\left(\frac{n\pi t}{L}\right)$ is even



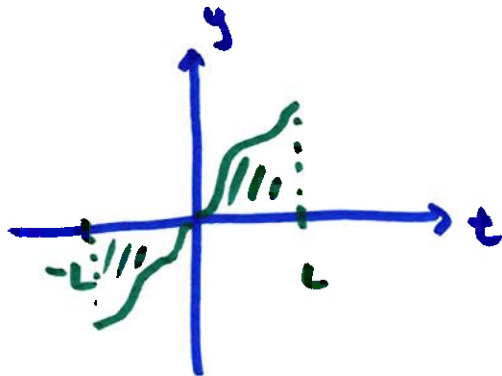
integral → area of shaded region

so, if $f(t)$ is even, then $a_n = \frac{2}{L} \int_{-L}^0 f(t) \cos\left(\frac{n\pi t}{L}\right) dt$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

even $f(t)$, $b_n = \frac{1}{L} \int_{-L}^L \underbrace{f(t)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}} dt$

odd



net area = 0 so if $f(t)$ is even,
 $b_n = 0$

if $f(t)$ is odd, following the same process, we find

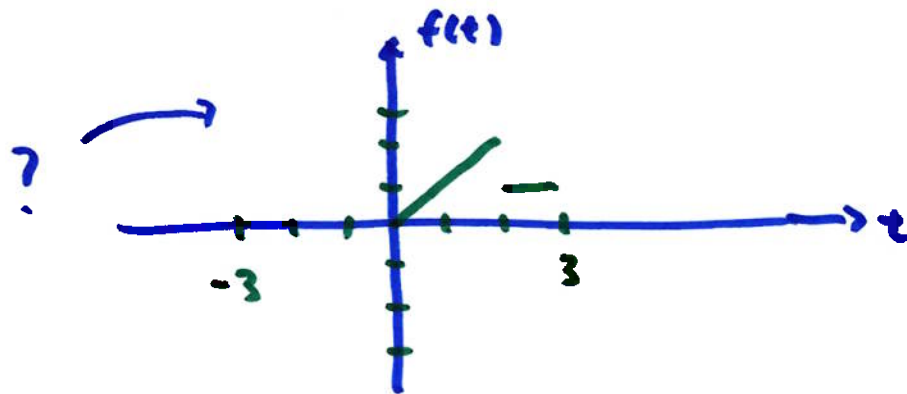
$$a_n = 0, \quad b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

$$f(t) \text{ even } : \left. \begin{aligned} a_n &= \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt \\ b_n &= 0 \rightarrow \text{no sine} \end{aligned} \right\} \text{Fourier Cosine Series}$$

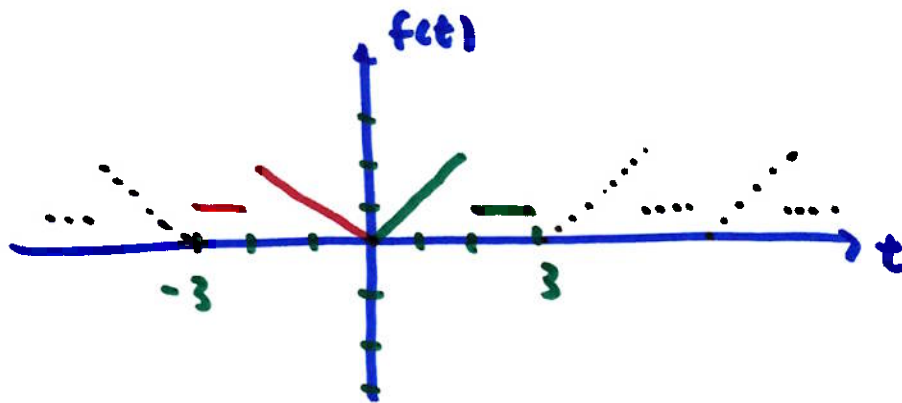
$$f(t) \text{ odd } : \left. \begin{aligned} a_n &= 0 \rightarrow \text{no cosine} \\ b_n &= \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt \end{aligned} \right\} \text{Fourier Sine Series}$$

if $f(t)$ with period $2L$ and given on $0 < t < L \rightarrow$ half the picture to complete the picture, we can choose even or odd extensions

$$f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 1 & \text{if } 2 < t < 3 \end{cases} \quad f(t) \text{ has period 6}$$

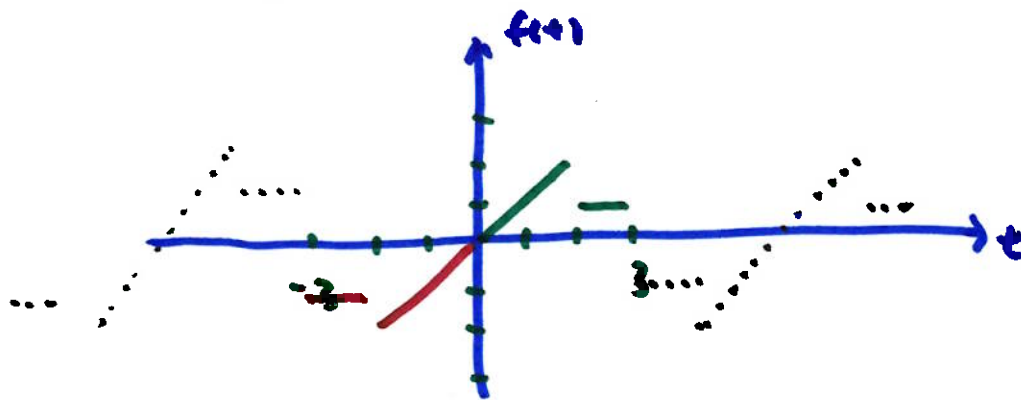


with even extensions $\rightarrow$ y-axis sym



$f(t)$ is now even
 $\rightarrow$ Cosine series

with odd extensions



$f(t)$ is now odd
 $\rightarrow$ Sine series

example

$$f(t) = \begin{cases} t & 0 < t < 2 \\ 1 & 2 < t < 3 \end{cases} \quad \text{period } 6$$

express as a Fourier Sine Series

↳ odd extensions

sine series $\rightarrow a_n = 0$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt \quad L: \text{half period}$$

$$= \frac{2}{3} \int_0^2 t \sin\left(\frac{n\pi t}{3}\right) dt + \frac{2}{3} \int_2^3 1 \cdot \sin\left(\frac{n\pi t}{3}\right) dt$$

$$= \dots = \frac{6 \sin\left(\frac{2n\pi}{3}\right) - 4n\pi \cos\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} + \frac{2 \cos\left(\frac{2n\pi}{3}\right) - 2 \cos(n\pi)}{n\pi}$$

$$\sin\left(\frac{2n\pi}{3}\right) = \begin{cases} \sqrt{3}/2 & n \text{ odd} \\ -\sqrt{3}/2 & n \text{ even} \\ 0 & n = 3k \end{cases} \quad \cos\left(\frac{2n\pi}{3}\right) = \begin{cases} -\frac{1}{2} & n \neq 3k \\ 1 & n = 3k \end{cases}$$

very difficult to express as summation

$\rightarrow$ write a few non-zero terms

result:

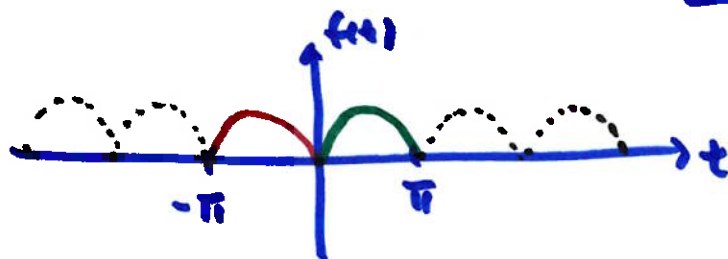
$$f(t) \sim \left(\frac{3\sqrt{3}}{\pi^2} + \frac{3}{\pi} \right) \sin\left(\frac{\pi t}{3}\right) + \left(\frac{-3\sqrt{3}}{4\pi^2} - \frac{1}{\pi} \right) \sin\left(\frac{2\pi t}{3}\right) \\ + \left(\frac{3\sqrt{3}}{16\pi^2} - \frac{1}{4\pi} \right) \sin\left(\frac{4\pi t}{3}\right) + \dots$$

example

$$f(t) = \sin(t) \quad 0 < t < \pi \quad \text{period } 2\pi$$

express as a Fourier cosine series

→ even extensions



$$\text{even} \rightarrow b_n = 0$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt \quad L = \pi$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(t) \cos(nt) dt$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(t) dt = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(t) \cos(nt) dt = \frac{2(\cos(n\pi) + 1)}{\pi(1-n^2)} = \begin{cases} \frac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

first few terms

$$\frac{2}{\pi} - \frac{4}{3\pi} \cos(2t) - \frac{4}{15\pi} \cos(4t) - \frac{4}{35\pi} \cos(6t) - \dots$$

~~not~~ a function doesn't have to be even or odd, it can be neither