9.3 Fourier Cosine and Sine series (part 2)

$f(t)$ periodic period $2L$

even: Fourier cosine series $b_n = 0$ $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

odd: Fourier sine series $a_n = 0$ $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

Boundary-value problems: conditions specified at boundaries instead of initial points (initial-value problems)

example: $x'' + 4x = 0$ $\quad x(\underset{=}{0}) = x(\underset{=}{\pi}) = 0$ w/o derivatives

boundaries

"Dirichlet condition"

NOT both at 0

(e.g. $x(0) = 0$, $x'(0) = 0$)

or $x'(0) = x'(\pi) = 0$ both derivatives

"Neumann condition"

"old way" solve $x'' + 4x = 0 \rightarrow r = \pm 2i$

$x(t) = C_1 \cos 2t + C_2 \sin 2t$, then use BC's to find C's

nonhomogeneous is more complicated

$$x'' + 2x = 1$$

"old way" solve homogeneous $x'' + 2x = 0$

then solve for the particular solution

due to nonhomogeneous term (undetermined coeffs or variation of params)

alternative way: use Fourier series (sines and cosines are very good at meeting BC's because they are periodic)

expand the nonhomogeneous term as Fourier series

expand x as Fourier series w/ unknown coeffs

then solve for the coeffs

example $x'' + 2x = 1 \quad x(0) = x(\pi) = 0$

first, expand the nonhomogeneous term $f(t) = 1$

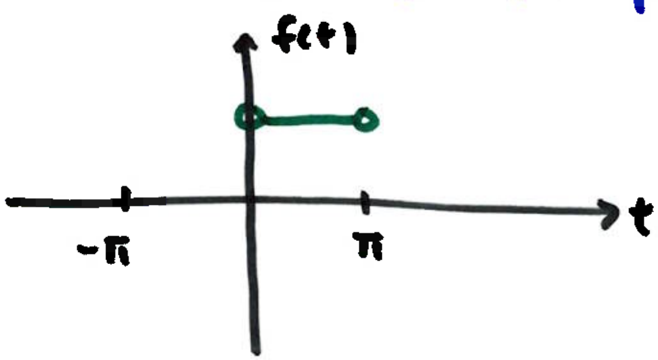
as a Fourier series

$f(t) = 1 \qquad 0 < t < \pi$

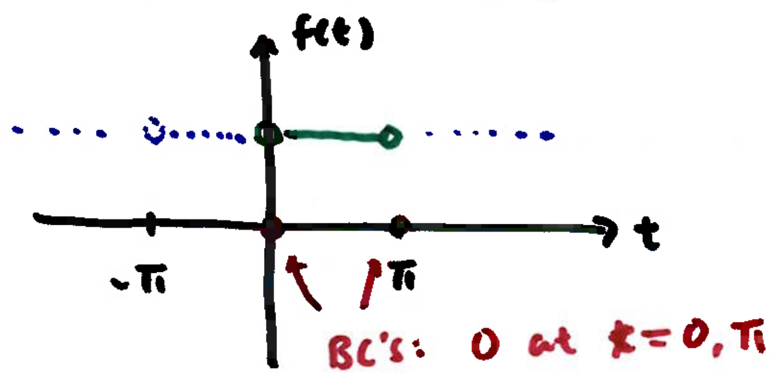
add either odd or even extensions whichever

meet the BC's $\quad x(0) = x(\pi) = 0$

$f(t) = 1 \quad 0 < t < \pi \quad$ period 2π

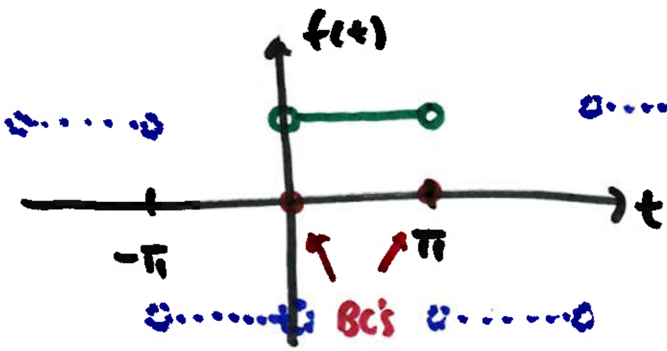


with even extensions



note w/ even extensions
the expansion will
NOT meet the BC's

with odd extensions



meet the BC's ?

yes. Because the series converges to the average value at discontinuities

so, we need to expand $f(t) = 1 \quad 0 < t < \pi$ period 2π

as a Fourier Sine series

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \cdots = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$f(t) = \sum_{n=0}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt)$$

now left side of $x'' + 2x = 1$

expand x as a general (unknown) Fourier series w/ same period as the right side

$$x(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

π π

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nt) + B_n \sin(nt)$$

Sub into left side of $x'' + 2x = 1$

$$x' = \sum_{n=1}^{\infty} -nA_n \sin(nt) + nB_n \cos(nt)$$

$$x'' = \sum_{n=1}^{\infty} -n^2 A_n \cos(nt) - n^2 B_n \sin(nt)$$

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt)$$

goal: find A_n and B_n

$$\underbrace{\sum_{n=1}^{\infty} -n^2 A_n \cos(nt) - n^2 B_n \sin(nt)}_{x''} + \underbrace{A_0 + \sum_{n=1}^{\infty} 2A_n \cos(nt) + 2B_n \sin(nt)}_{2x}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt)}_{1}$$

collect like terms

$$A_0 + \sum_{n=1}^{\infty} (2-n^2) A_n \cos(nt) + \sum_{n=1}^{\infty} \underline{(2-n^2) B_n} \sin(nt) = \sum_{n=1}^{\infty} \underbrace{\underline{\frac{2}{n\pi}\left[1-(-1)^n\right]}}_{\text{NO constants, NO cosines}} \sin(nt)$$

NO constants

NO cosines

so, $A_0 = 0$ $A_n = 0$

and $B_n = \frac{2}{n\pi(2-n^2)}\left[1-(-1)^n\right]$

so, the solution to $x'' + 2x = 1$ $x(0) = x(\pi) = 0$

is $$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi(2-n^2)}\left[1-(-1)^n\right] \sin(nt)}$$

first few terms:

$$x(t) = \frac{4}{\pi}\sin(t) - \frac{4}{21\pi}\sin(3t) - \frac{4}{105\pi}\sin(5t) - \cdots$$