9.4 Application of Fourier Series (part 2)

$m x'' + c x' + k x = F(t)$

example $x'' + 9x = F(t)$

no damping

$$F(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

$$= \cdots = \sum_{n=1}^{\infty} \frac{2}{n}\left[1-(-1)^n\right] \sin(nt)$$

let $x = \sum_{n=1}^{\infty} B_n \sin(nt)$

$x' = \cdots$

$x'' = \cdots$

Sub all into ODE above

the steady periodic solution is

$$x_{sp} = \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{\underbrace{n(9-n^2)}_{0 \text{ when } n=3}} \sin(nt)$$

amplitude $\to \infty$

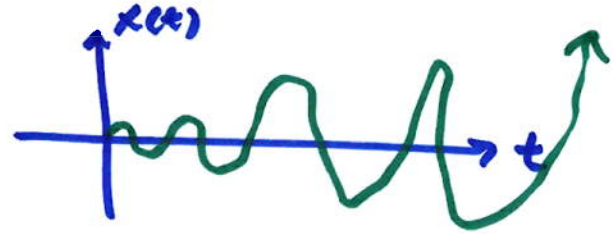
because $x'' + 9x = 0 \to x = C_1 \cos 3t + C_2 \sin 3t$

resonance

general solution: solve $x'' + 9x = A\sin 3t + B\sin 3t$ undetermined coeff

plus $x_{sp} = \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{2[1-(-1)^n]}{n(9-n^2)} \sin(nt)$

$$x(t) = C_1\cos 3t + C_2\sin 3t \underbrace{- \frac{2}{3}t\cos 3t}_{\text{resonance } \to \infty} + \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{2[1-(-1)^n]}{n(9-n^2)} \sin(nt)$$



one way to fix that is adding a damper ($c \neq 0$)

$mx'' + cx' + kx = F(t)$

assume $F(t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$

now expand x in Fourier series $x(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi t}{L}\right) + B_n\sin\left(\frac{n\pi t}{L}\right)$

$x' = \sum_{n=1}^{\infty} -A_n\frac{n\pi}{L}\sin\left(\frac{n\pi t}{L}\right) + B_n\frac{n\pi}{L}\cos\left(\frac{n\pi t}{L}\right)$

$x'' = \ldots$

sub into ODE above

$$x_{sp} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right) \qquad (A_0 = 0)$$

let $\omega_n = \frac{n\pi}{L}$, after simplification, we get

$$A_n = \frac{-F_n c\,\omega_n}{(k - m\omega_n^2)^2 + (c\omega_n)^2} \qquad B_n = \frac{F_n (k - m\omega_n^2)}{(k - m\omega_n^2)^2 + (c\omega_n)^2}$$

denom $\neq 0$ (amplitude does not $\to \infty$)

different form: rewrite $A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$

$$= C_n \sin(\omega_n t - \alpha_n)$$

phase shift

we get: $C_n = \dfrac{F_n}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}$ $\qquad \alpha_n = \tan^{-1}\left(\dfrac{c\,\omega_n}{k - m\omega_n^2}\right)$

damper gives phase shift

$$x_{sp} = \sum_{n=1}^{\infty} C_n \sin(\omega_n t - \alpha_n)$$

$$\omega_n = \frac{n\pi}{L}$$

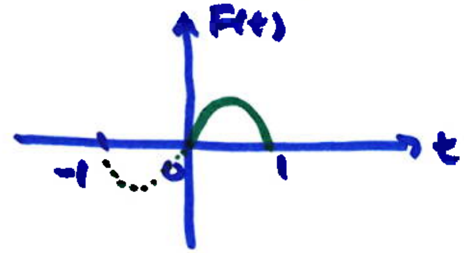
example $x'' + x' + 13x = F(t)$ $\quad F(t) = t - t^2 \quad 0<t<1 \quad$ period 2

homogeneous solution

$$x'' + x' + 13x = 0$$

$$x_h(t) = c_1 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{51}}{2} t\right) + c_2 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{51}}{2} t\right)$$

add odd extensions



$$F(t) = \cdots = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left[1-(-1)^n\right]\sin(n\pi t)$$

$x_h \to 0$ as $t \to \infty$

($c \neq 0 \to$ negative exponential)

$$x_{sp} = \cdots = \sum_{n=1}^{\infty} \frac{\frac{4}{n^3\pi^3}\left[1-(-1)^n\right]}{\sqrt{(13-(n\pi)^2)^2 + (n\pi)^2}} \sin\left(n\pi t - \tan^{-1}\left(\frac{n\pi}{13-n^2\pi^2}\right)\right)$$

$$= 0.0582 \sin(\pi t - 0.7872) + 0.000125 \sin(3\pi t - 3.0179) + 0.000009(5\pi t - 3.0745) + \cdots$$

dominant term: with $\sin(\pi t) \to$ period 2 from force

when $t \to \infty$, the dominant motion is $\sin(\pi t)$ term
which matches the input force

if force matches the frequency of the fundamental motion
(resonance), damper prevents amplitude from $\to \infty$
but the amplitude can still be large