9.5 Heat Equation and Separation of Variables

conducting (metal) rod length L

heated, then temperature at any point and at time t is $u(x,t)$

$u(x,t)$ obeys the 1-D Heat Equation

heat only flows along rod (no up/down/side flow)

constant ($k>0$)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_t = k u_{xx}$$

$0 \le x \le L \quad , \quad t > 0$

$u_t = k u_{xx}$ is a partial differential equation (PDE)

↳ partial derivatives ($mx'' + cx' + kx = f(t)$ is ODE)

this PDE has two boundary conditions

$u(0,t) = T_1$ temp at left end ($x=0$)

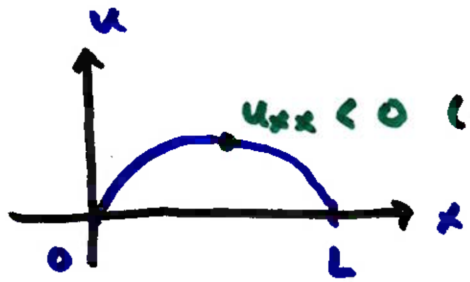
$u(L,t) = T_2$ temp at right end ($x=L$)

if $T_1 = T_2 = 0$ then the ~~equation~~ BC's are said to be homogeneous

one initial condition

$u(x,0) = f(x)$ initial temp. profile ($t=0$)

what does $u_t = k u_{xx}$ say?



$u_{xx} < 0$ (concave down)

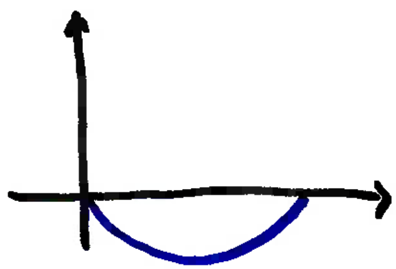
if $u_{xx} < 0$ at some x, t

then $u_t = k u_{xx} < 0$

$u_t < 0 \rightarrow$ heat is flowing away (temp $\downarrow$)

$u_{xx} < 0 \rightarrow$ temp. > avg. of nearby

heat goes hot to cool



$u_{xx} > 0$ $\quad$ $u_t = k u_{xx} > 0$ heat flows in

temp < avg. temp nearby

if $u_{xx} = 0 \rightarrow u_t = 0$

Solution method: separation of variables

$u_t = k u_{xx}$ BC's: $u(0,t) = u(L,t) = 0$ (homogeneous)

IC: $u(x,0) = f(x)$

$u(x,t) = X(x)\,T(t)$ product of one function of x and one of t

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial}{\partial t}\left(X(x)T(t)\right) = X(x)\frac{\partial}{\partial t}\left(T(t)\right) = X T'$$

$$\frac{\partial u}{\partial x} = u_x = X'T$$

$$u_{xx} = X''T$$

$$u_t = k u_{xx}$$

$$XT' = k X''T$$

$$\frac{X''}{X} = \frac{T'}{kT} = \text{constant} = -\lambda \quad (\lambda > 0)$$ separation constant

Solve $\frac{X''}{X} = -\lambda \rightarrow \boxed{X'' + \lambda X = 0}$ this is now ODE

BC's: $u(0,t) = X(0)T(t) = 0 \rightarrow \boxed{X(0)=0}$ $\quad X'' + \lambda X = 0$

$u(L,t) = X(L)T(t) = 0 \rightarrow \boxed{X(L)=0}$

solution of $X'' + \lambda X = 0$ (think $x'' + 4x = 0$)

$$X = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$X(0) = 0 \rightarrow c_1 = 0$$

$$X(L) = 0 \rightarrow c_2 \sin(\sqrt{\lambda}\, L) = 0 \qquad c_2 \neq 0$$

$$\sin(\sqrt{\lambda}\, L) = 0$$

$$\sqrt{\lambda}\, L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$ eigenvalues

fundamental solution: $X = \sin(\sqrt{\lambda}\, x)$

$$= \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{X_n = \sin\left(\frac{n\pi x}{L}\right)}$$ eigenfunctions

now $\frac{T'}{kT} = -\lambda = -\frac{n^2\pi^2}{L^2}$

$$T' + \frac{kn^2\pi^2}{L^2} T = 0 \rightarrow T(t) = d_1 e^{-\frac{kn^2\pi^2}{L^2}t}$$

for each n, $\boxed{T_n = e^{-\frac{kn^2\pi^2 t}{L^2}}}$

$$u(x,t) = X T$$

so for each n, $u_n = T_n X_n = e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$

general solution: sum all up

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

IC: $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$ Sine series

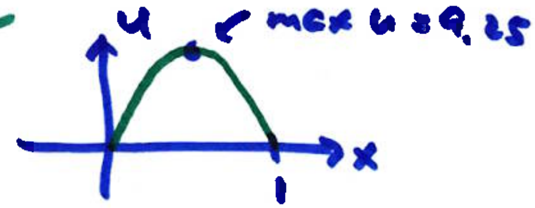
$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

__example__ $u_t = 2u_{xx}$ $\quad 0 < x < 1$ $\leftarrow L = 1$ $, t > 0$

$\nearrow k = 2$

$u(0,t) = u(1,t) = 0$ both ends at temp. $= 0$

$u(x,0) = \underbrace{9\sin(\pi x) - \frac{1}{4}\sin(3\pi x)}_{f(x)}$



$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2\pi^2 t} \sin(n\pi x)$$

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

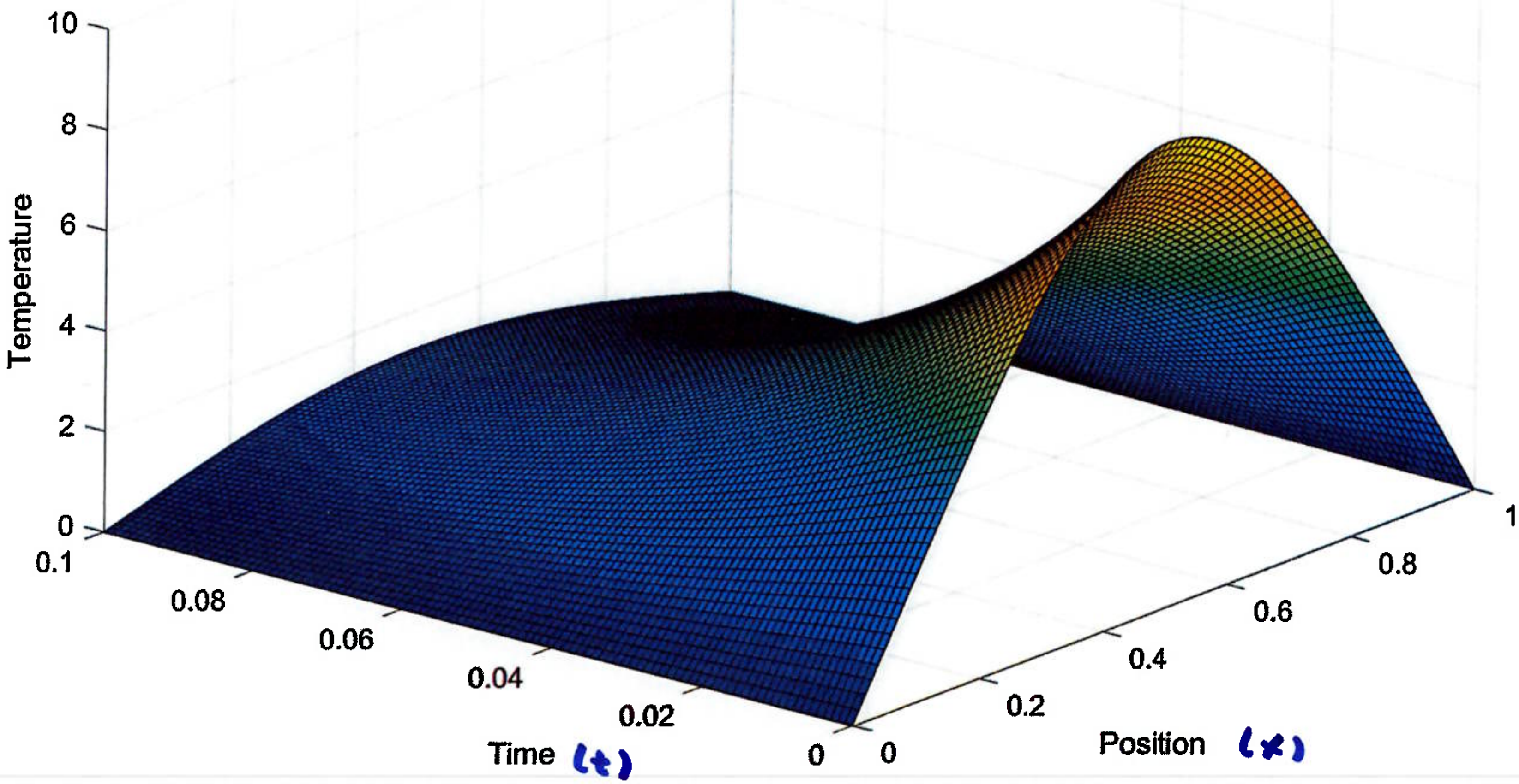
for this one, since $f(x) = 9\sin(\pi x) - \frac{1}{4}\sin(3\pi x)$ is already a Fourier series

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = 9\sin(\pi x) - \frac{1}{4}\sin(3\pi x)$$

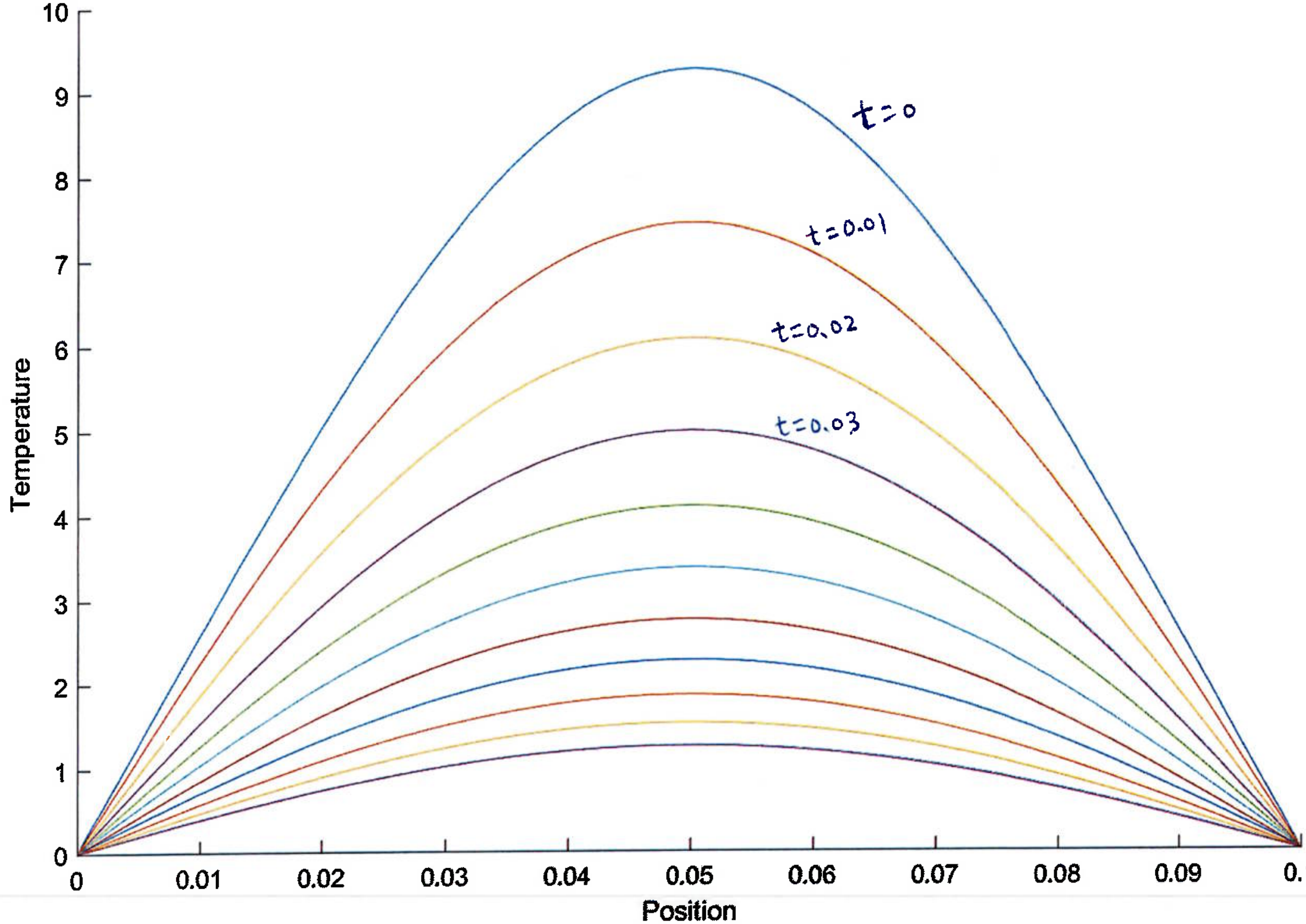
$b_1 = 9,\quad b_3 = -\frac{1}{4}$ all others zero

$$\boxed{u(x,t) = 9e^{-2\underset{\uparrow 1}{n}^2\pi^2 t}\sin(\pi x) - \frac{1}{4}e^{-18\pi^2 t}\sin(3\pi x)}$$

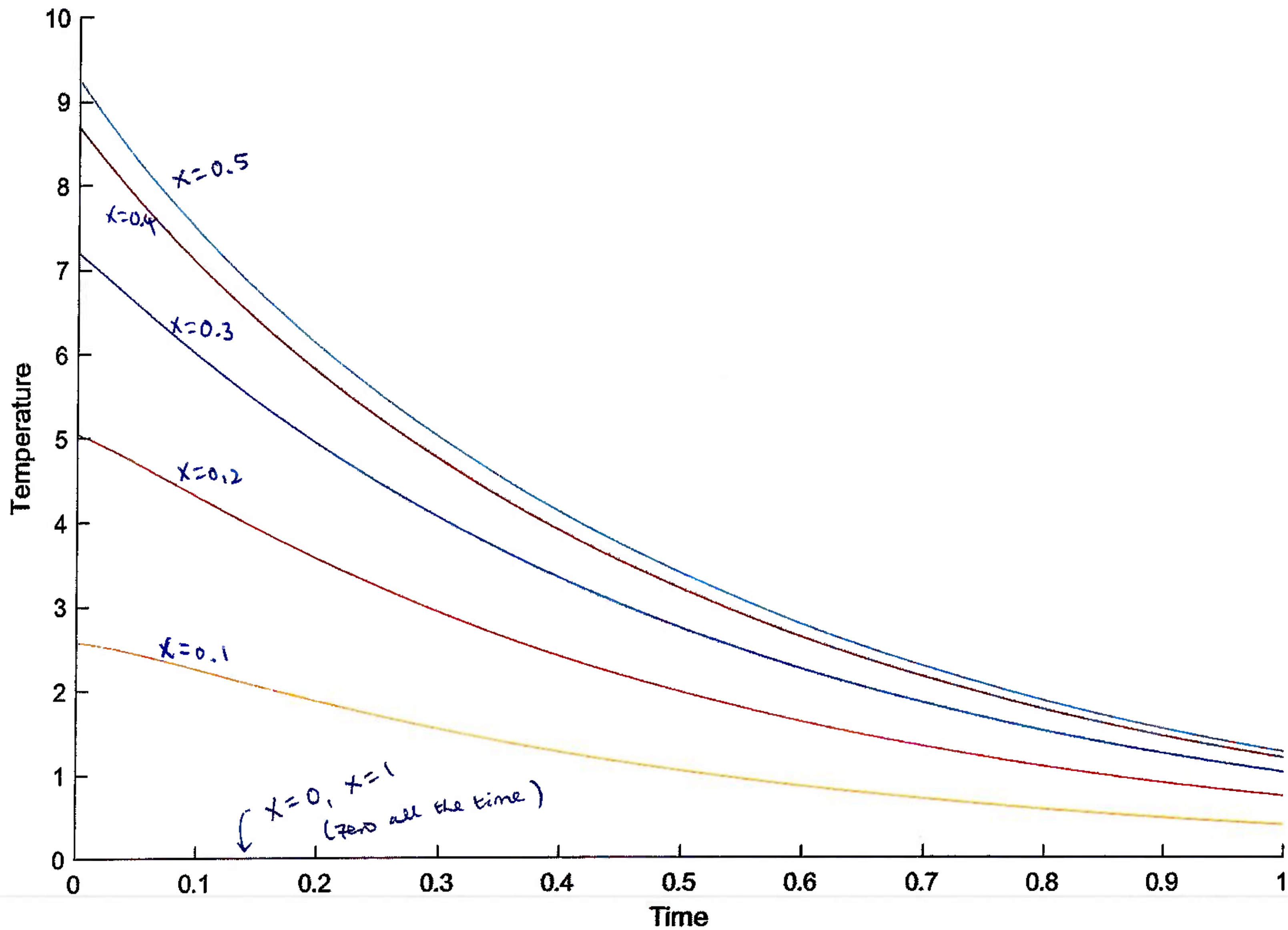
↳ surface



10
8
6
4
2
0
Temperature
0.1
0.08
0.06
0.04
0.02
0
Time (t)
0
0.2
0.4
0.6
0.8
1
Position (x)



t=0
t=0.01
t=0.02
t=0.03
Temperature
Position
0
1
2
3
4
5
6
7
8
9
10
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1



Temperature
Time
10
9
8
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x=0.5
x=0.4
x=0.3
x=0.2
x=0.1
x=0, x=1
(zero all the time)