9.6 Wave Equation (part 1)

string length L

ends fixed

pluck it

y : string displacement from equilibrium

$y=0$ equilibrium

$y>0$: above equilibrium

$y<0$: below "

goal: find $y(x,t)$

the governing equation: Wave Equation

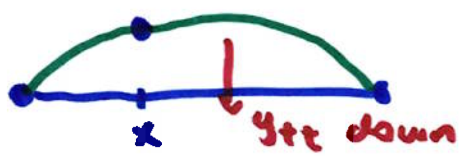
$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad y_{tt} = a^2 y_{xx}$$

$$a^2 = \frac{\text{tension}}{\text{density}}$$

physical meaning of $y_{tt} = a^2 y_{xx}$

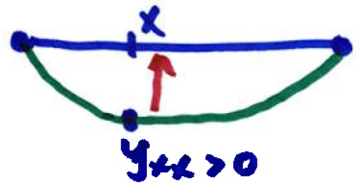
if $y_{xx} < 0$ at some x

$y_{xx} < 0$ concave down



$a^2 > 0$ so if $y_{xx} < 0$ then $y_{tt} < 0 \rightarrow$ downward acceleration

if $y_{xx} > 0$



$y_{xx} > 0 \rightarrow y_{tt} > 0$ upward acc

if $y_{xx} = 0$

$y_{tt} = 0 \rightarrow$ no acceleration

$y_{tt} = a^2 y_{xx}$

just like heat eq, there are boundary conditions (BC's) and initial conditions (IC's)

BC's: $y(0,t) = y(L,t) = 0$ end positions fixed at 0 (Dirichlet condition)

$y_t(0,t) = y_t(L,t) = 0$ end velocities 0 (Neumann condition)

IC's: $y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ initial velocity

Problem A: $f(x) \neq 0$ $g(x) = 0$ displacement only

Problem B: $f(x) = 0$ $g(x) \neq 0$ velocity only

first, we solve problem A

$$y_{tt} = a^2 y_{xx} \qquad y(0,t) = y(L,t) = 0 \quad \text{(BC's)}$$

$$y(x,0) = f(x) \quad \text{(IC)}$$

$$y_t(x,0) = 0$$

Same method as in heat eq. → separation of variables

$$y(x,t) = X(x)\,T(t)$$

$$y_t = XT' \qquad y_{tt} = XT''$$

$$y_x = X'T \qquad y_{xx} = X''T$$

Sub into $y_{tt} = a^2 y_{xx}$

$$XT'' = a^2 X''T$$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda = \text{separation constant (same as in heat eq.)}$$

$$\frac{X''}{X} = -\lambda \rightarrow \boxed{X'' + \lambda X = 0}$$

BC's: $y(0,t) = 0$

$X(0)T(t) = 0 \rightarrow \boxed{X(0) = 0}$

$y(L,t) = 0$

$X(L)T(t) = 0 \rightarrow \boxed{X(L) = 0}$

exactly the same as in heat eq.

eigenvalues $\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$ $n = 1, 2, 3, \ldots$

eigenfunctions $\boxed{X_n = \sin\left(\frac{n\pi x}{L}\right)}$

time problem: $\frac{T''}{a^2 T} = -\lambda = -\frac{n^2\pi^2}{L^2}$

$$\boxed{T'' + \frac{n^2\pi^2 a^2}{L^2} T = 0}$$

IC: $y_t(x,0) = 0$

$X T'(0) = 0 \rightarrow \boxed{T'(0) = 0}$

$\vdots$

$$\boxed{T_n = \cos\left(\frac{n\pi a t}{L}\right)}$$

$y(x,t) = X(x)T(t)$

for each n, $y_n = X_n T_n = \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, 3, \ldots$

general solution: linear combo for all n

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

IC: $y(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

sine series $\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$