9.6 Wave Equation (part 2)

$y_{tt} = a^2 y_{xx}$

$y(0,t) = y(L,t) = 0$

$y(x,0) = f(x)$

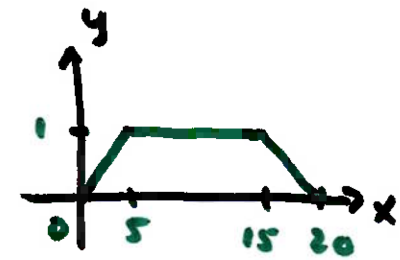
$y_t(x,0) = g(x)$

Problem A : $g(x) = 0, \quad f(x) \neq 0$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

example : $L = 20, \quad a = 1, \quad f(x) = \begin{cases} \frac{1}{5}x & 0 \le x \le 5 \\ 1 & 5 \le x \le 15 \\ \frac{20-x}{5} & 15 \le x \le 20 \end{cases}$



$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{20}\right) \sin\left(\frac{n\pi x}{20}\right)$$

$$A_n = \frac{2}{20} \int_0^{20} f(x) \sin\left(\frac{n\pi x}{20}\right) dx$$

$$= \frac{2}{20}\left[\int_0^5 \frac{1}{5}x \sin\left(\frac{n\pi x}{20}\right) dx + \int_5^{15} \sin\left(\frac{n\pi x}{20}\right) dx + \int_{15}^{20} \frac{20-x}{5} \sin\left(\frac{n\pi x}{20}\right) dx\right]$$

$$= \frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right) \qquad n = 1, 2, 3, 4, \ldots$$

n=1: Fundamental Mode (first harmonic / overtone)

$$y_1(x,t) = \frac{8\sqrt{2}}{\pi^2} \cos\left(\frac{\pi}{20} t\right) \sin\left(\frac{\pi}{20} x\right)$$

↳ fundamental circular frequency

$$\frac{\pi/20}{2\pi} = \frac{1}{40} \text{ Hz}$$

if this were a string of an instrument, we would hear sound of frequency $\frac{1}{40}$ Hz

fundamental mode is often the most dominant one

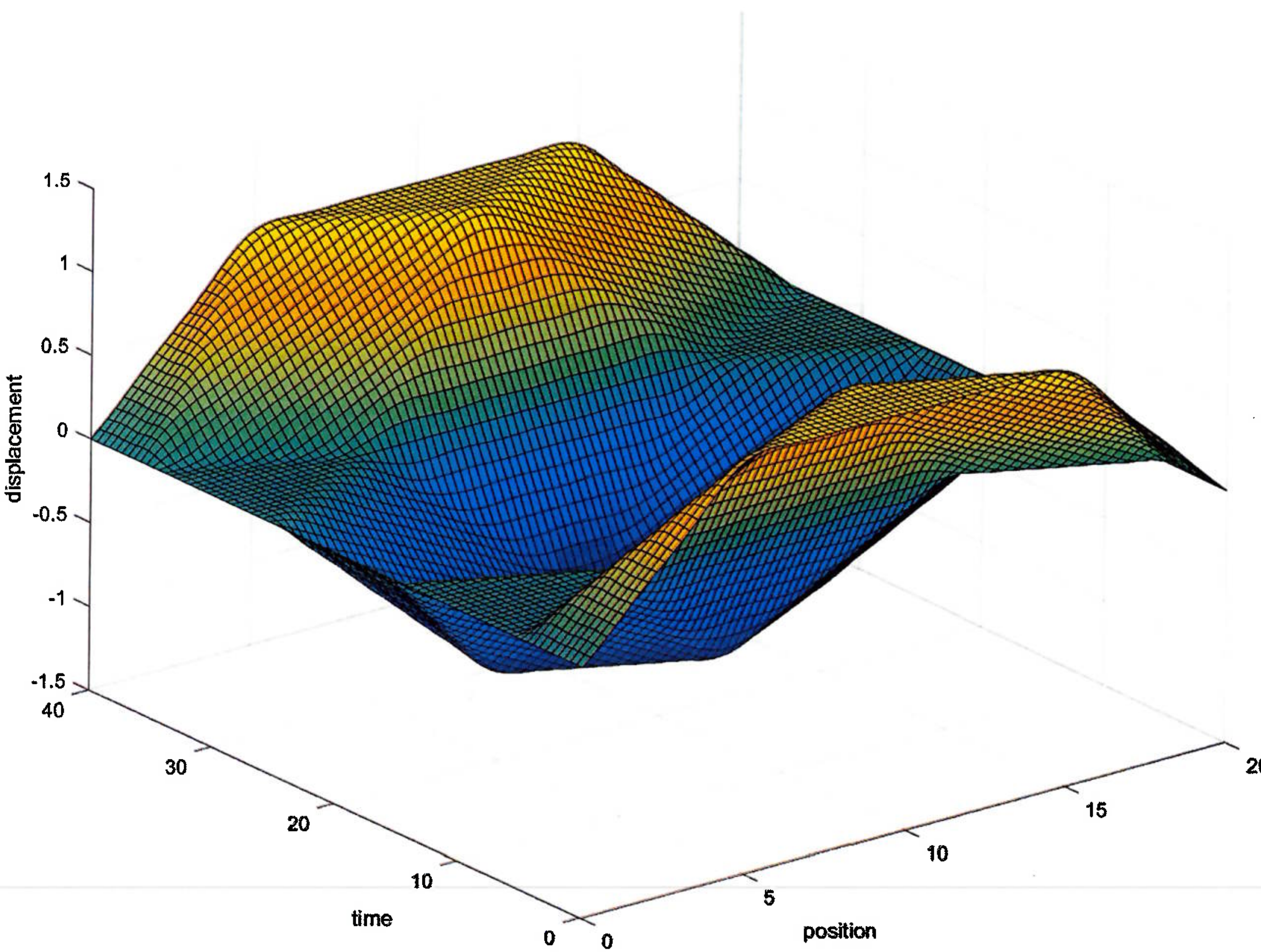
n=2: second harmonic / overtone

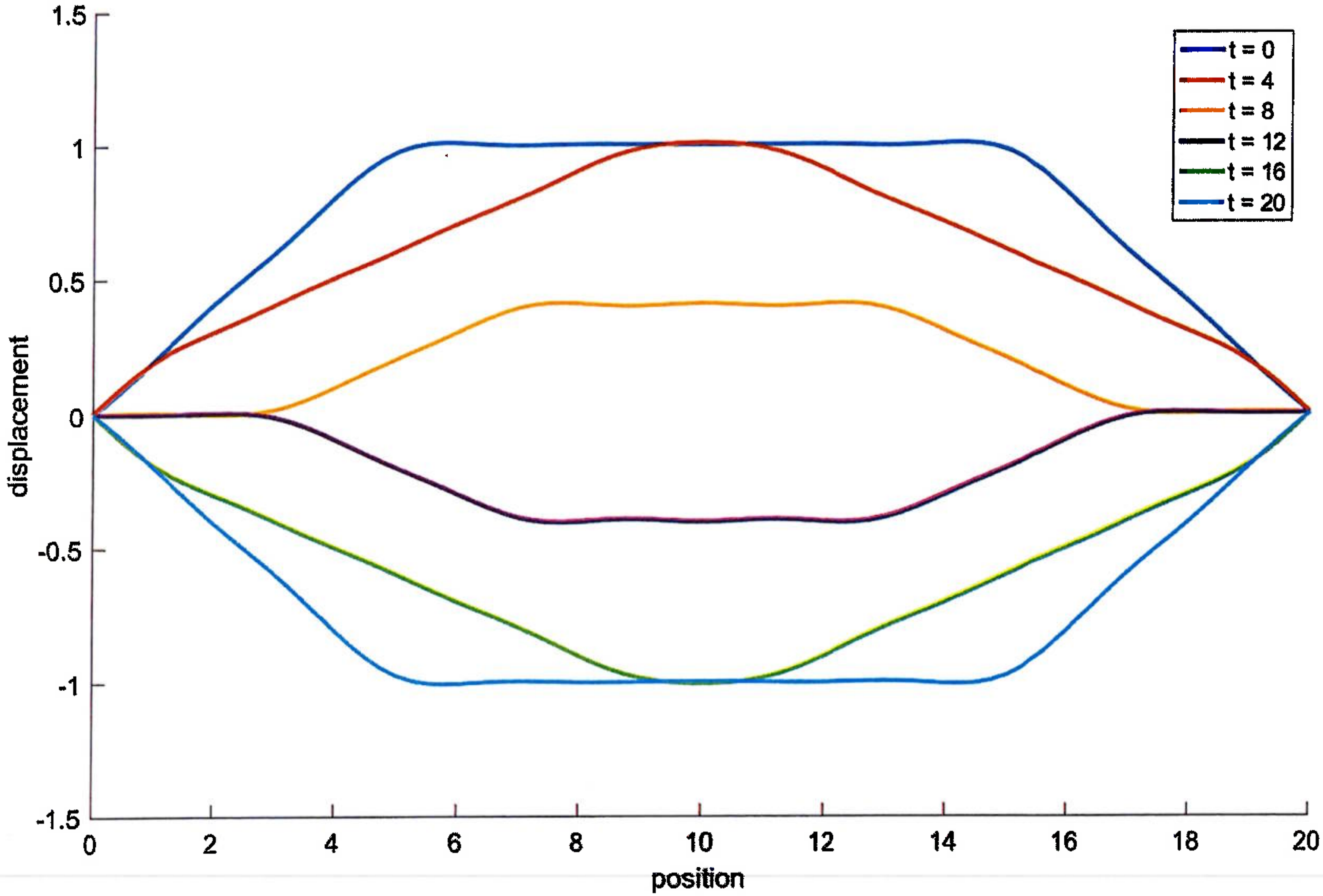
frequency: $\frac{n\pi}{20} = 2 \cdot \frac{\pi}{20}$ doubling the first harmonic

Sound is one octave higher

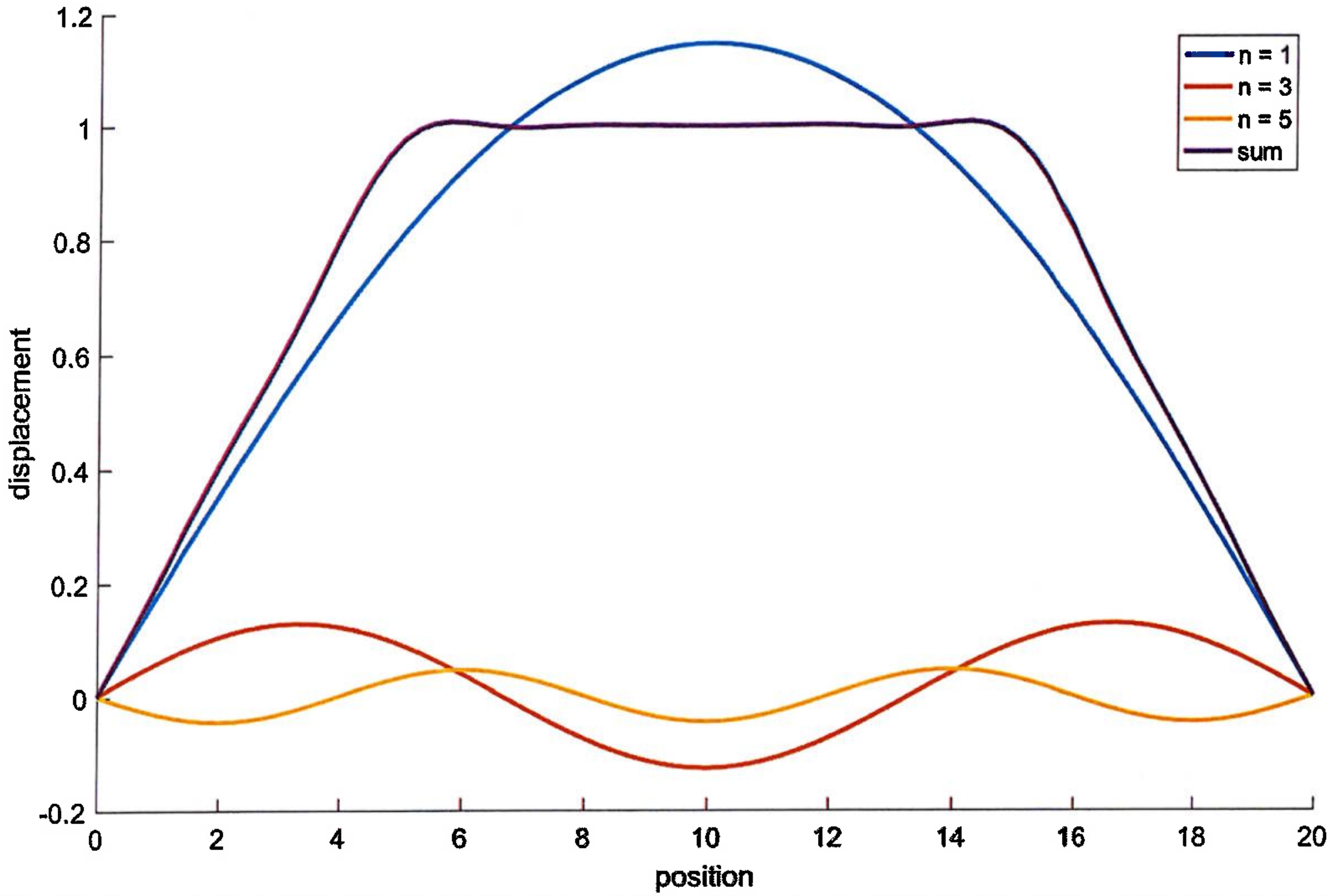
n=3: third harmonic

one octave plus a perfect fifth above fundamental


displacement
1.5
1
0.5
0
-0.5
-1
-1.5
time
40
30
20
10
0
position
0
5
10
15
20



1.5
1
0.5
0
-0.5
-1
-1.5
displacement
0
2
4
6
8
10
12
14
16
18
20
position
t = 0
t = 4
t = 8
t = 12
t = 16
t = 20



n = 1
n = 3
n = 5
sum
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
displacement
0
2
4
6
8
10
12
14
16
18
20
position

Problem B: $y_{tt} = a^2 y_{xx}$

$y(0,t) = y(L,t) = 0$

$y(x,0) = 0$

$y_t(x,0) = g(x)$

$y = X(x)T(t) \qquad y_{tt} = XT'' \qquad y_{xx} = X''T$

$XT'' = a^2 X''T$

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

BC's: $y(0,t)=0 \rightarrow X(0)T(t)=0 \rightarrow X(0)=0$

$y(L,t)=0 \rightarrow X(L)T(t)=0 \rightarrow X(L)=0$

Space part is identical to Problem A

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad X_n = \sin\left(\frac{n\pi x}{L}\right)$$

time problem: $T'' + a^2\lambda T = 0$ IC: $y(x,0)=0$

$$T'' + \frac{a^2n^2\pi^2}{L^2}T = 0 \qquad X(x)T(0)=0 \rightarrow T(0)=0$$

$$T = C_1\cos\left(\frac{n\pi a t}{L}\right) + C_2\sin\left(\frac{n\pi a t}{L}\right)$$

$T(0)=0 \rightarrow 0 = C_1$ so, $T_n = \sin\left(\frac{n\pi a t}{L}\right)$

so, $y_n = \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$

Prob. A: $y_n = \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

unused IC: $y_t(x,0) = g(x)$

$$y_t(x,t) = \sum_{n=1}^{\infty}\left(B_n \cdot \frac{n\pi a}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

at $t=0$

$$g(x) = \sum_{n=1}^{\infty}\left(B_n \cdot \frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Sine series w/ coeff $B_n \cdot \frac{n\pi a}{L}$

$$B_n \cdot \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

general solution: $\underbrace{f(x) \neq 0}_{A}$, $\underbrace{g(x) \neq 0}_{B}$

solve A ignoring $g(x)$, solve B ignoring $f(x)$

then add them together