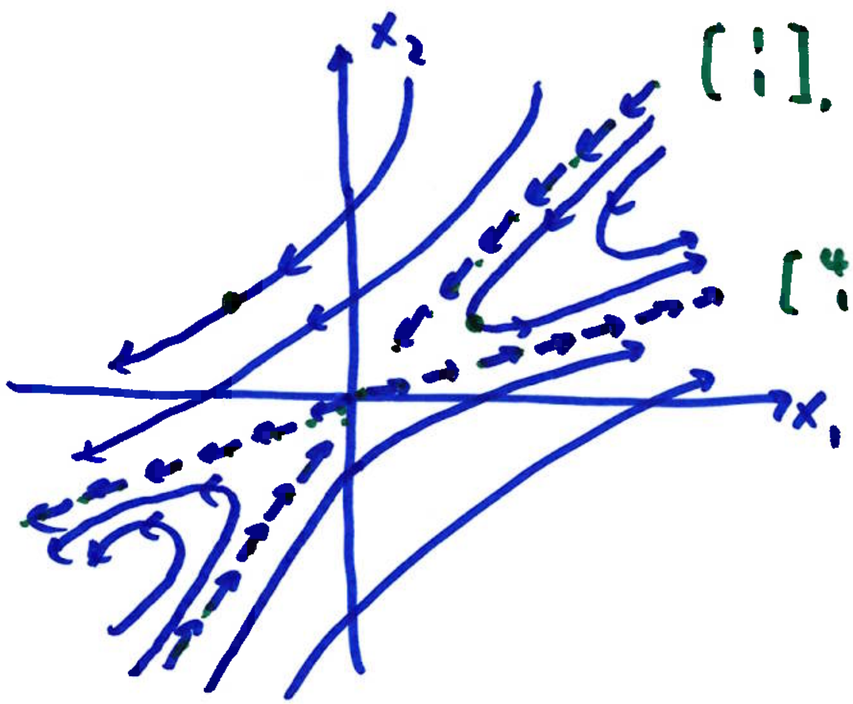
5.3 Solution Curves of Linear Systems

different eigenvalue combos → different curves

<u>example</u> $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \vec{x}$

$\left.\begin{array}{l} \lambda_1 = 2, \quad \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}\right\} \vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda < 0$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\lambda > 0$

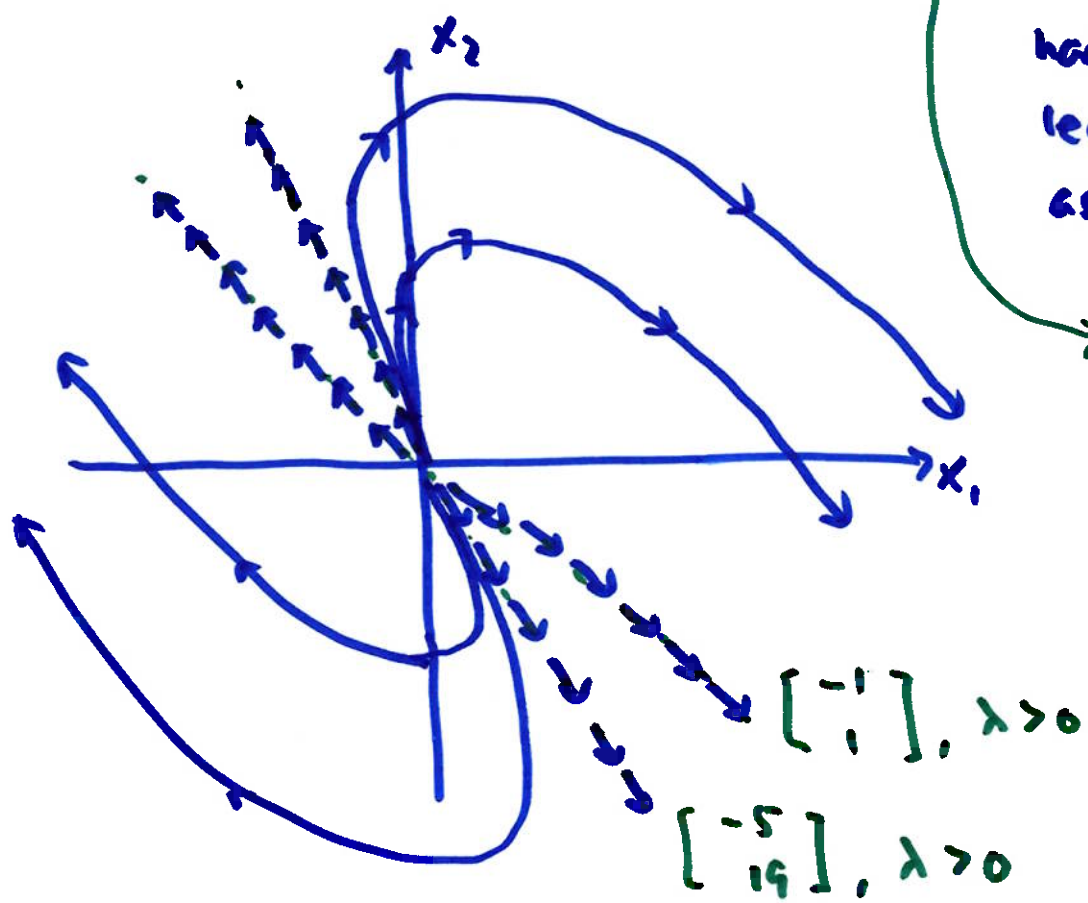
origin is a <u>saddle point</u>

λ's opposite in signs

example

$\vec{x}' = \begin{bmatrix} 22 & 5 \\ -19 & -2 \end{bmatrix} \vec{x}$

$\left.\begin{array}{l} \lambda_1 = 3, \quad \vec{v}_1 = \begin{bmatrix} -5 \\ 19 \end{bmatrix} \\ \lambda_2 = 17, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}\right\} \vec{x}(t) = c_1 e^{3t} \begin{bmatrix} -5 \\ 19 \end{bmatrix} + c_2 e^{17t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



harder to see how solutions leaving origin will go → which asymptote to follow?

when $t \to \infty$

$e^{17t} >> e^{3t}$

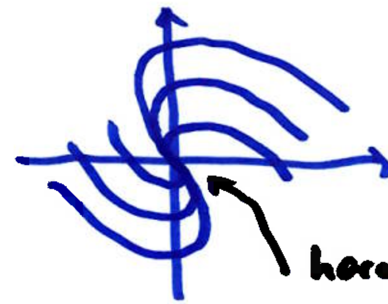
so $t \to \infty$ means follow $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

so, leaving origin we follow $\begin{bmatrix} -5 \\ 19 \end{bmatrix}$

the origin here is called an improper source

sink if both $\lambda < 0$

"improper" because multiple solutions follow the same asymptote leaving or entering origin



hard to tell solutions apart

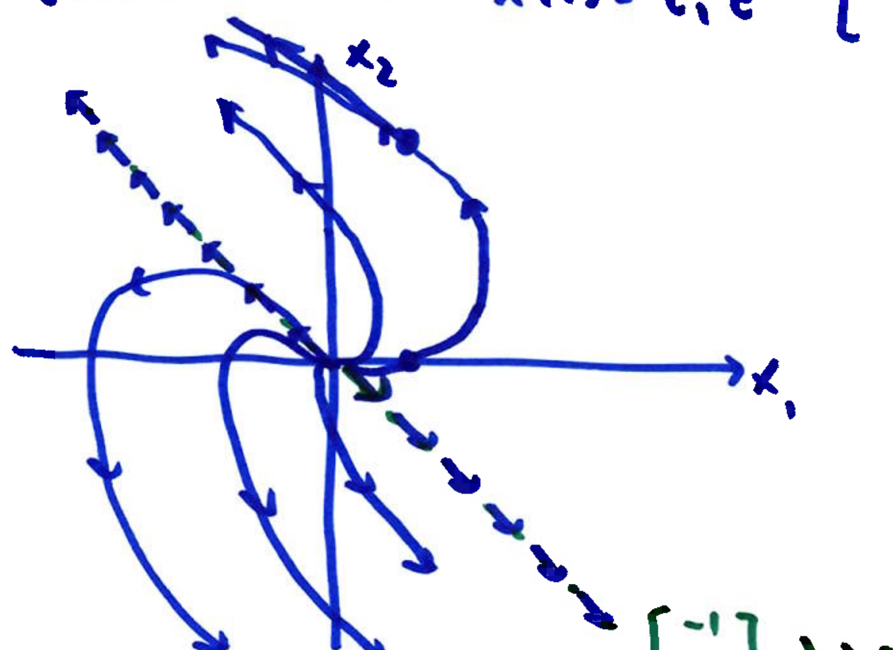
<u>improper</u> source/sink → λ's same sign

<u>example</u> $\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \vec{x}$

$\lambda = 4, 4$ only one "real" eigenvector $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

general solution $\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \vec{v}_2 \right)$

can't see it



again, improper node (source here)

use the matrix to help visualization $\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \vec{x}$

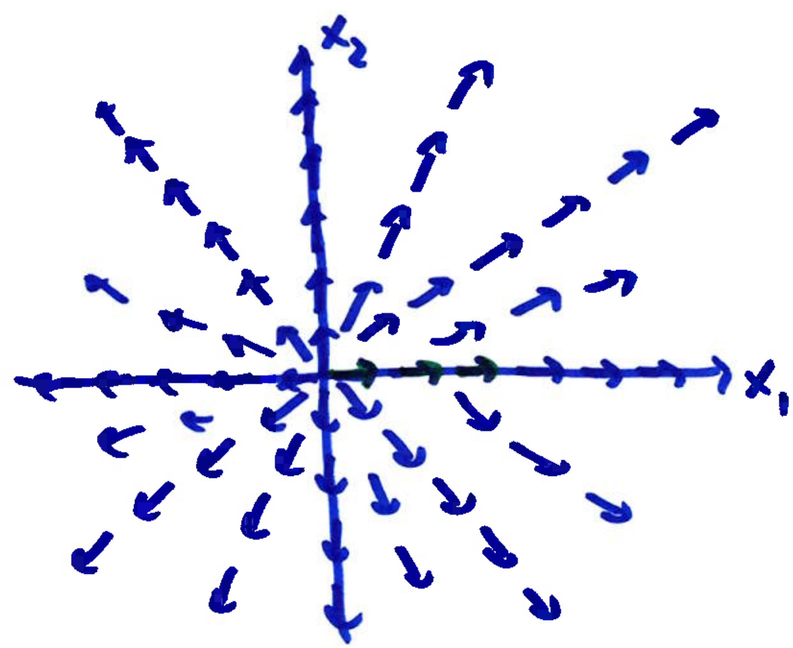
choose $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x} \rightarrow \vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow$ right 1, up 3

$\begin{bmatrix} 1 \\ 10 \end{bmatrix} = \vec{x} \rightarrow \vec{x}' = \begin{bmatrix} -29 \\ 77 \end{bmatrix}$

__improper source/sink__ also when λ's repeated, A defective

__example__ $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad$ A is complete



use $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ for the rest

if $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} a \\ b \end{bmatrix}$

__proper source__

repeated λ's, complete A

complex: spirals out if $Re(\lambda) > 0$

spirals in if $Re(\lambda) < 0$

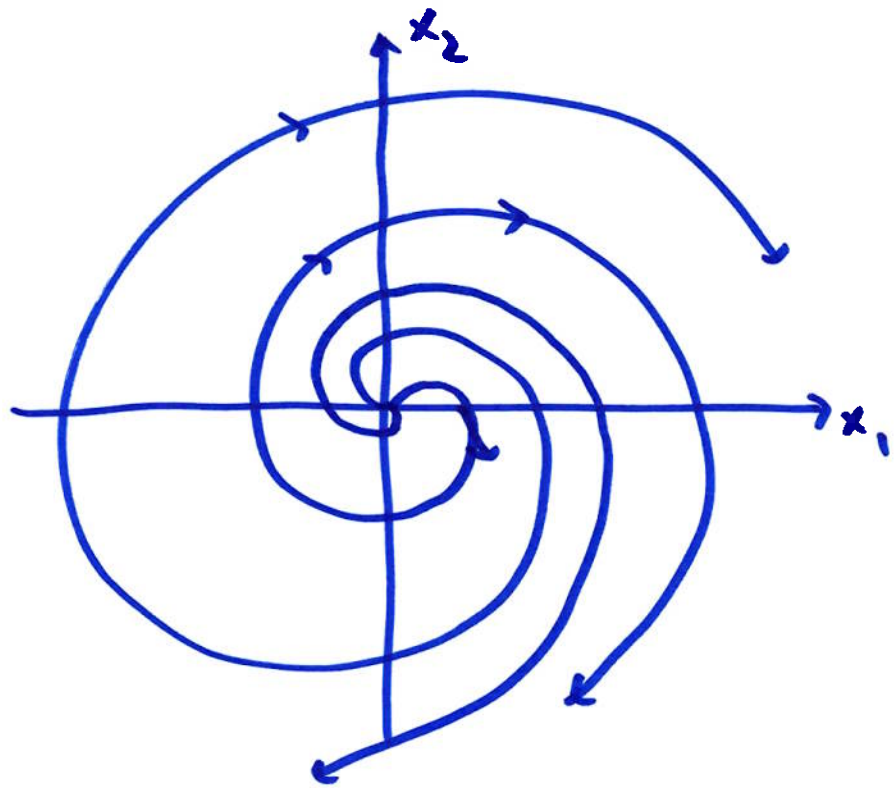
ovals if $Re(\lambda) = 0$

<u>example</u> $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

$\lambda_1 = 1 - i, \quad \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = 1 + i, \quad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\vec{x}(t) = c_1 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$



here, spirals <u>out</u>

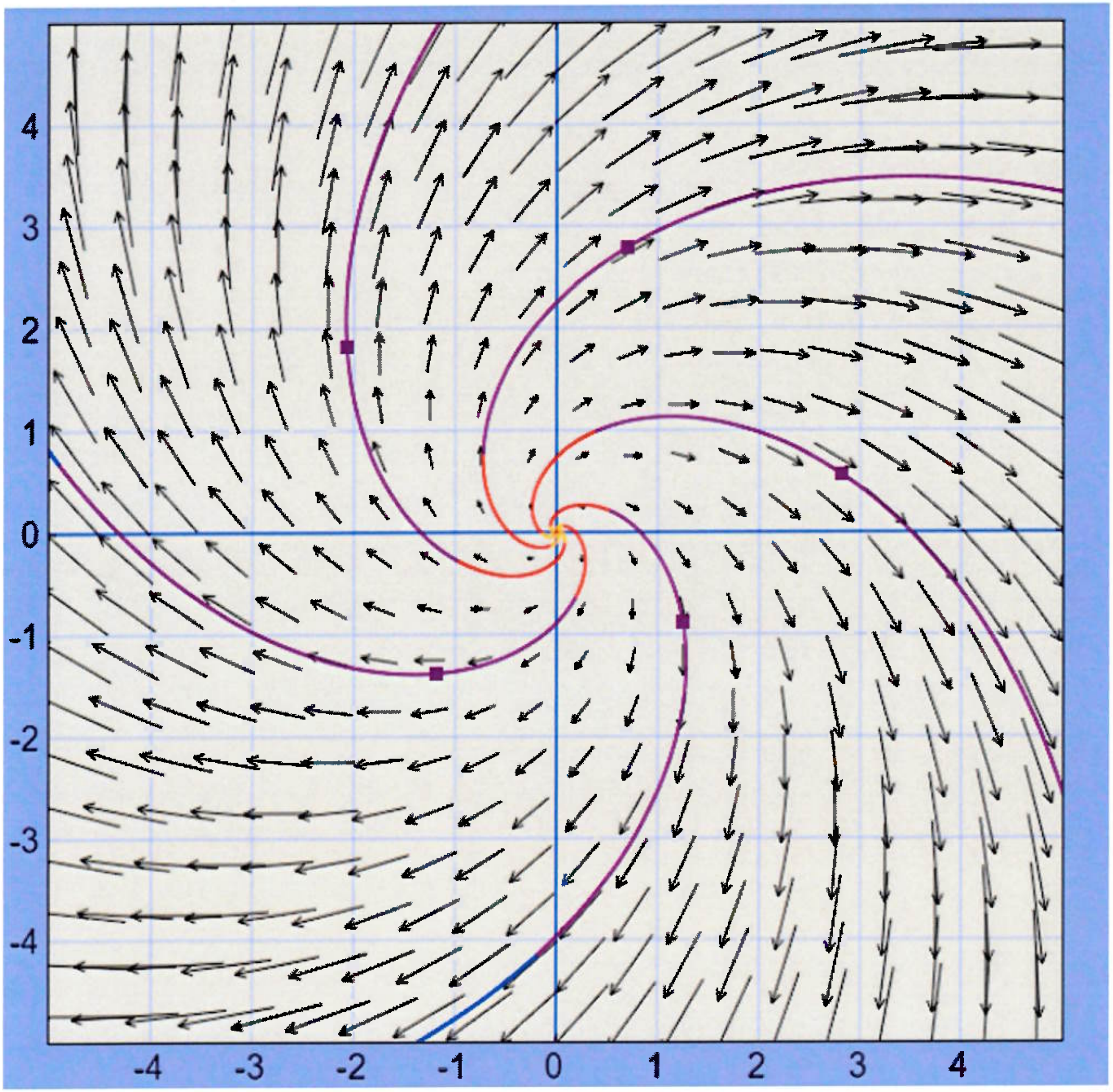
but cw or ccw?

back to $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}\,' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

right 1, down 1

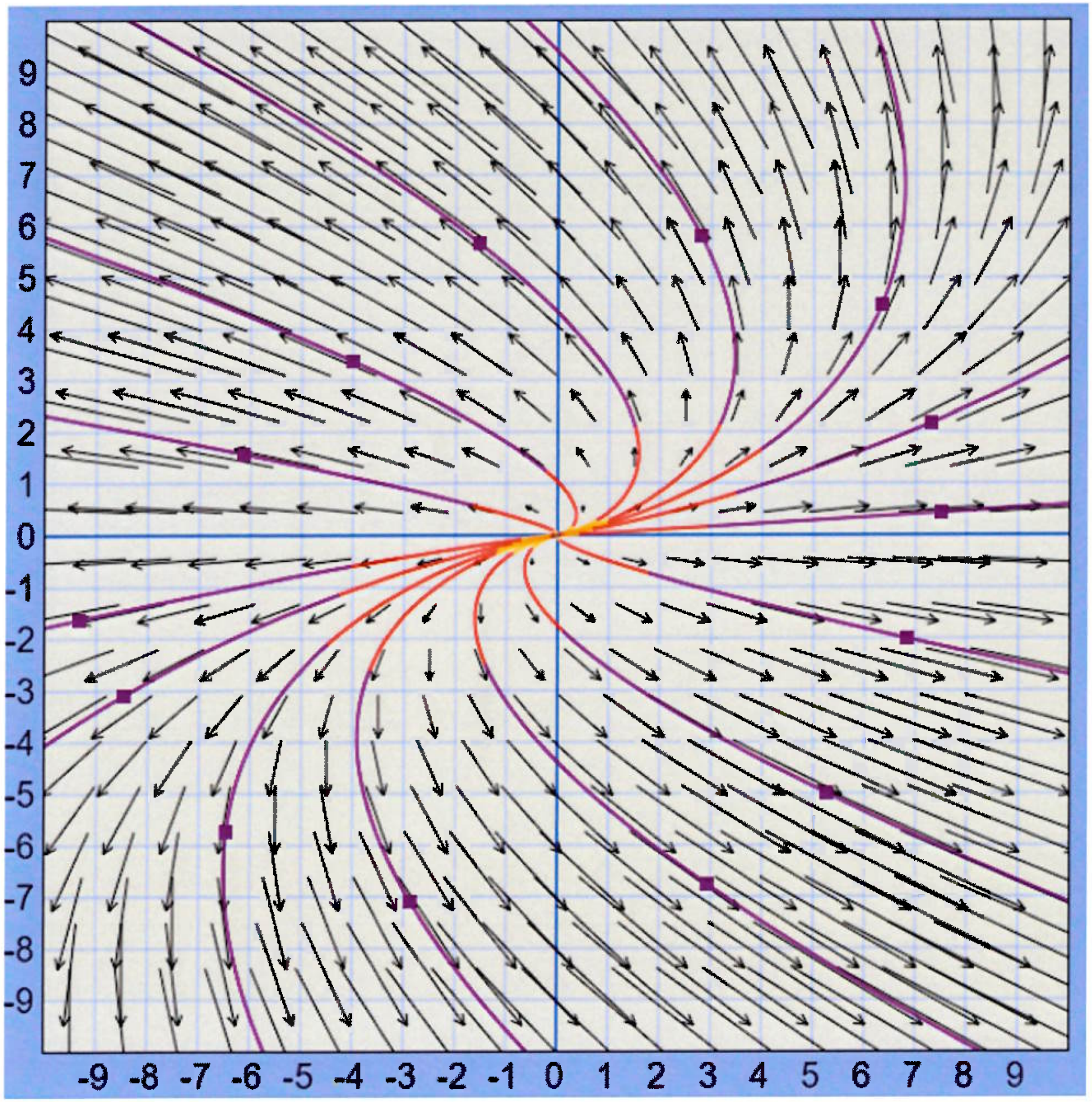
so cw



λ's complex

$Re(\lambda) > 0$

can't see eigenvectors
in picture



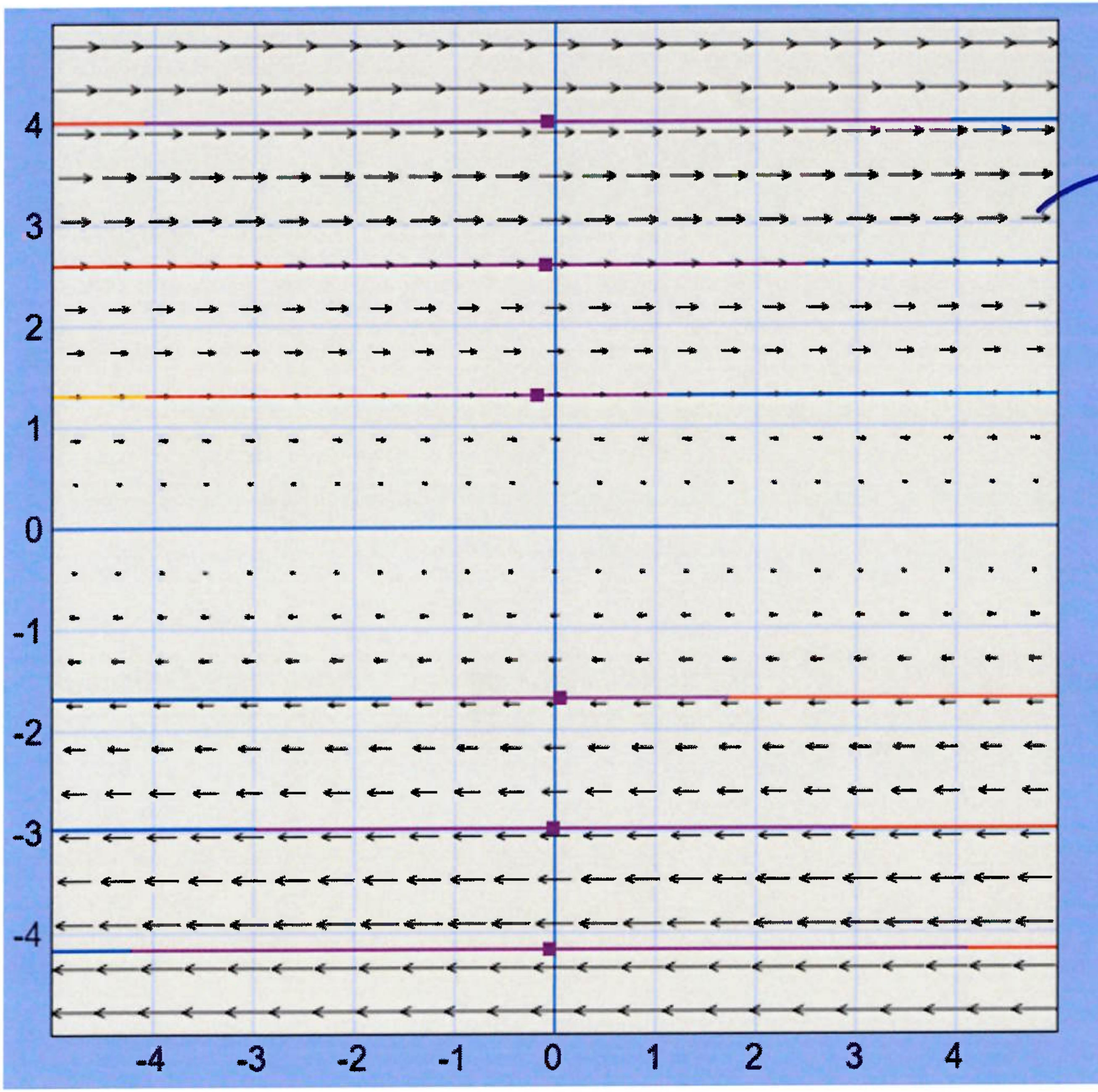
λ: real, positive (both)

straight line: $\begin{bmatrix}1\\0\end{bmatrix}$

so one eigenvector is $\begin{bmatrix}1\\0\end{bmatrix}$

no other straight lines that solutions use as asymptotes

so, probably repeated λ w/ defective A



$\vec{x}(t) = \cancel{\begin{bmatrix} c_1 \\ 3 \end{bmatrix}}$

$= \begin{bmatrix} a+bt \\ 3 \end{bmatrix}$

$x_2 = \text{constant} = C_2$

$\lambda = 0$

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$

$\lambda = 0, 0$