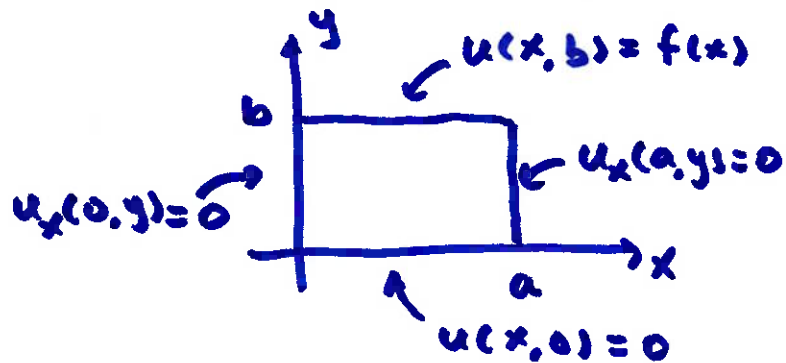


9.7 Laplace's Eq. (part 2)

$$u_{xx} + u_{yy} = 0 \quad 0 < x < a \quad 0 < y < b$$

there is a variety of BC's



$$\left. \begin{array}{l} u(x,0) = 0 \\ u_x(0,y) = 0 \\ u_x(a,y) = 0 \\ u(x,b) = f(x) \end{array} \right\} \begin{array}{l} \text{homogeneous, used to} \\ \text{solve } X, Y \end{array}$$

nonhomogeneous, used at the end for coefficients of series

$$u(x,y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = C$$

$$u_{xx} = X''Y \quad u_{yy} = XY''$$

C depends on BC's

$$u(x,0) = 0 \rightarrow X(x)Y(0) = 0 \rightarrow Y(0) = 0$$

$$u_x(0,y) = 0 \rightarrow X'(0)Y(y) = 0 \rightarrow X'(0) = 0$$

$$u_x(a,y) = 0 \rightarrow X'(a)Y(y) = 0 \rightarrow X'(a) = 0$$

X has two BC's $\rightarrow$ solve for X first

$$X'' - cX = 0 \quad X'(0) = X'(a) = 0$$

check $c = 0 \rightarrow$ nontrivial X that meets BC's

$$X'' = 0 \rightarrow X = c_1 x + c_2$$

$$X' = c_1 \quad X'(0) = X'(a) = 0 \rightarrow c_1 = 0$$

$$\text{so, } \boxed{X = 1} \quad \text{if } c = 0 \rightarrow \boxed{\lambda = 0}$$

the BC's $X'(0) = X'(a) = 0$ suggest sine and cosine

so $c < 0 \rightarrow$ let $c = -\lambda$

$$X'' + \lambda X = 0 \quad X'(0) = X'(a) = 0$$

seen this before:

$$\boxed{\lambda_n = \frac{n^2 \pi^2}{a^2}}$$

$$\boxed{X_n = \cos\left(\frac{n\pi x}{a}\right)}$$

$$n = 1, 2, 3, \dots$$

also good for $\boxed{n = 0, 1, 2, 3, \dots}$

and that includes $c = 0$ case

now solve $-\frac{Y''}{Y} = C = -\lambda = -\frac{n^2\pi^2}{a^2} \quad n=0,1,2,3,\dots$

find Y when $n=0$

$$Y''=0 \rightarrow Y = C_1 y + C_2 \quad \text{BC: } Y(0)=0$$

$$0 = C_2 \quad \text{so, } \boxed{Y = y} \quad \text{when } \boxed{\lambda = n = 0}$$

$$n > 0 \quad \frac{Y''}{Y} = \frac{n^2\pi^2}{a^2}$$

$$Y'' - \frac{n^2\pi^2}{a^2} Y = 0 \quad Y(0) = 0$$

$$Y = C_1 \cosh\left(\frac{n\pi y}{a}\right) + C_2 \sinh\left(\frac{n\pi y}{a}\right)$$

$$0 = C_1$$

$$\boxed{Y_n = \sinh\left(\frac{n\pi y}{a}\right)} \quad n=1,2,3,\dots$$

$$u_n = \sum_n Y_n$$

for $n=0$, $u_0 = (1)(y) = y$

for $n=1,2,3,\dots$ $u_n = \sinh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right)$

$$u(x, y) = \underbrace{\frac{1}{2} C_0 y}_{\text{choose } \frac{1}{2} C_0 \text{ for coeff for } n=0 \text{ because we will end up w/ a cosine series}} + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right)$$

choose $\frac{1}{2} C_0$ for coeff for $n=0$ because we will end up w/ a cosine series

last BC: $u(x, b) = f(x)$

$$f(x) = \underbrace{\frac{1}{2} [C_0 b]}_{\text{"a}_0\text{"}} + \sum_{n=1}^{\infty} \underbrace{\left[C_n \sinh\left(\frac{n\pi b}{a}\right) \right]}_{\text{"a}_n\text{"}} \cos\left(\frac{n\pi x}{a}\right)$$

cosine series w/ coeff
"a₀" and "a_n"

cosine series coeff: $\frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{a}\right) dx$
 $\swarrow$
 L for x is a

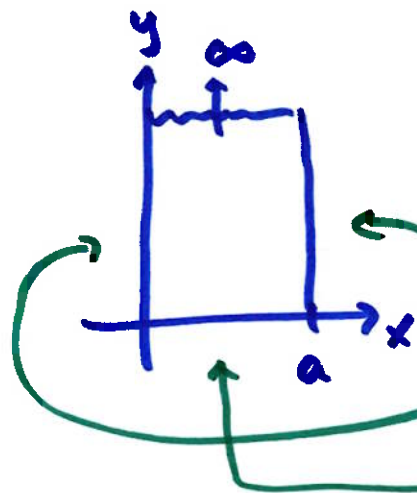
$$a_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$a_0 = \frac{2}{a} \int_0^a f(x) dx$$

$$C_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$C_0 = \frac{2}{ab} \int_0^a f(x) dx$$

we can model a very long plate by using semi-infinite domain



$$0 < x < a$$

let $y \rightarrow \infty$ impose some kind of BC there
as an example, let's insulate left and right

$$u_x(0, y) = 0$$

$$u_x(a, y) = 0$$

$$u(x, 0) = f(x)$$

u is bounded as $y \rightarrow \infty$ this is a BC at ∞

$$u = XY$$

X solution is identical to previous example

$$X_n = \cos\left(\frac{n\pi x}{a}\right)$$

$$\lambda_n = \frac{n^2\pi^2}{a^2}$$

$$n = 0, 1, 2, 3, \dots$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$Y'' - \lambda Y = 0 \rightarrow Y'' - \frac{n^2\pi^2}{a^2} Y = 0$$

$$Y = c_1 e^{n\pi y/a} + c_2 e^{-n\pi y/a}$$

exponentials are more convenient when dealing w/ ∞

BC: u bounded as $y \rightarrow \infty$ $u = \sum Y$ $\rightarrow Y$ bounded as $y \rightarrow \infty$
so, $C_1 = 0$ $\boxed{Y_n = e^{-n\pi y/a}}$ $n = 0, 1, 2, 3, \dots$

$$u = \sum Y$$

$$u_0 = 1 \quad (X_0 \text{ and } Y_0 \text{ are both } 1)$$

$$u_n = e^{-n\pi y/a} \cos\left(\frac{n\pi x}{a}\right)$$

$$\boxed{u(x, y) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-n\pi y/a} \cos\left(\frac{n\pi x}{a}\right)}$$

$$\text{last BC: } u(x, 0) = f(x)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right) \quad \text{cosine series}$$

$$\boxed{a_0 = \frac{2}{a} \int_0^a f(x) dx}$$

$$\boxed{a_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx}$$