10.1 Sturm-Liouville Problem (part 1)

in Ch. 9 we looked at problems of the type

$\mathbf{X}'' + \lambda \mathbf{X} = 0$ subject to $\mathbf{X}(0) = \mathbf{X}(L) = 0$ (e.g. heat temp = 0 at ends)

or $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$ (e.g. heat w/ insulated ends)

or $\mathbf{X}(0) = \mathbf{X}'(L) = 0$

or $\mathbf{X}'(0) = \mathbf{X}(0) = 0$

in solving heat, wave, Laplace's eq., we discovered that

$\mathbf{X}'' + \lambda \mathbf{X} = 0$ w/ $\mathbf{X}(0) = \mathbf{X}(L) = 0 \rightarrow \lambda_n = \frac{n^2\pi^2}{L^2}$ $\quad \mathbf{X}_n = \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, 3$

eigenvalue $\qquad$ eigenfunction

$\mathbf{X}'(0) = \mathbf{X}'(L) = 0 \rightarrow \lambda_n = \frac{n^2\pi^2}{L^2}$ $\quad \mathbf{X}_n = \cos\left(\frac{n\pi x}{L}\right)$

$n = 0, 1, 2, 3, \ldots$

based on BC's, λ and $\mathbf{X}$ behave a certain way

$y'' + \lambda y = 0 \quad 0<x<L$ is a special case of the

Sturm-Liouville Problem

negative

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0 \qquad a<x<b$$

negative

BC's: $\alpha_1 y(a) - \alpha_2 y'(a) = 0$

$\beta_1 y(b) + \beta_2 y'(b) = 0$

positive

$\Bigg\}$ α_1, α_2 cannot be both zero
β_1, β_2 " " " "

for example, $y'' + \lambda y = 0 \quad 0<x<L$

$y(0) = 0$

$y(L) = 0$

$\Bigg\}$ equivalent to $X'' + \lambda X = 0$

$X(0) = X(L) = 0$

(space problem from heat w/ end temp $= 0$)

notice this is a Sturm-Liouville (SL) problem

w/ $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

$\alpha_1 = 1$, $\alpha_2 = 0$

$\beta_1 = 1$, $\beta_2 = 0$

in SL problem, if $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous on $[a,b]$ and if $p(x)>0$ and $r(x)>0$ on $[a,b]$, the SL problem is said to be <u>regular</u>.

if the SL problem is regular, then the eigenvalues λ form an increasing sequence with a minimum value but no maximum value

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty \quad \text{no max}$$

minimum (which means can't be $-\infty$)

furthermore, if $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the BC's can be made nonnegative, then the λ's are also nonnegative.

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for example, $y''+\lambda y=0 \qquad 0<x<L$

$y(0)=0 \to \alpha_1 y(a)-\alpha_2 y'(a)=0 \to \alpha_1=1,\ \alpha_2=0$

$y(L)=0 \to \beta_1 y(b)+\beta_2 y'(b)=0 \to \beta_1=1,\ \beta_2=0$

$\to$ regular SL, nonnegative α, β, $\to \lambda \geq 0$

$y'' + \lambda y = 0 \qquad 0 < x < L$

can be made nonnegative w/o changing BC

$y'(0) = 0 \to \alpha_1 y(a) - \alpha_2 y'(a) = 0 \to \alpha_1 = 0, \ \alpha_2 = 1$

$y'(L) = 0 \to \beta_1 = 0, \ \beta_2 = 1$

$\to$ regular SL, nonnegative $\alpha, \beta \to \lambda \geq 0$

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) - y'(0) = 0 \to \alpha_1, \alpha_2 = 1$

$y(L) = 0 \longrightarrow \beta_1 = 1, \ \beta_2 = 0$

$\alpha, \beta \geq 0 \to \lambda \geq 0$

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) + y'(0) = 0 \to \alpha_1 = 1, \ \alpha_2 = -1$

$y(L) = 0 \longrightarrow \beta_1 = 1, \ \beta_2 = 0$

regular, but there is a ~~non~~ negative α or β

$\to$ negative λ is possible

example $y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0$

$hy(L) + y'(L) = 0 \qquad h > 0$

↳ can model how right end of a rod is insulated based on its temperature

identify α's and β's

$$\left.\begin{array}{ll} \alpha_1 = 1, & \alpha_2 = 0 \\ \beta_1 = h, & \beta_2 = 1 \end{array}\right\} \text{ since } h>0, \text{ all } \lambda\text{'s are } \geq 0$$

must consider
$\lambda = 0$ and $\lambda > 0$
(but not $\lambda < 0$)

<u>example</u> $y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0$

$h y(L) - y'(L) = 0 \qquad h > 0$

$\alpha_1, \alpha_2 \geq 0$

$\beta_1 = h > 0, \quad \beta_2 < 0$

must consider $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

Start with $\lambda < 0$

for convenience, $\lambda = -K^2 \qquad K > 0$

$y'' - K^2 y = 0$

$y = C_1 \cosh(Kx) + C_2 \sinh(Kx)$

$y(0) = 0 \rightarrow C_1 = 0 \rightarrow y = C_2 \sinh(Kx)$

$y' = C_2 K \cosh(Kx)$

$h y(L) - y'(L) = 0$

$h C_2 \sinh(KL) - C_2 K \cosh(KL) = 0$

$C_2 \neq 0$
so y is non-trivial
→ whole point of Sturm-Liouville prob

$h \sinh(KL) - K \cosh(KL) = 0$

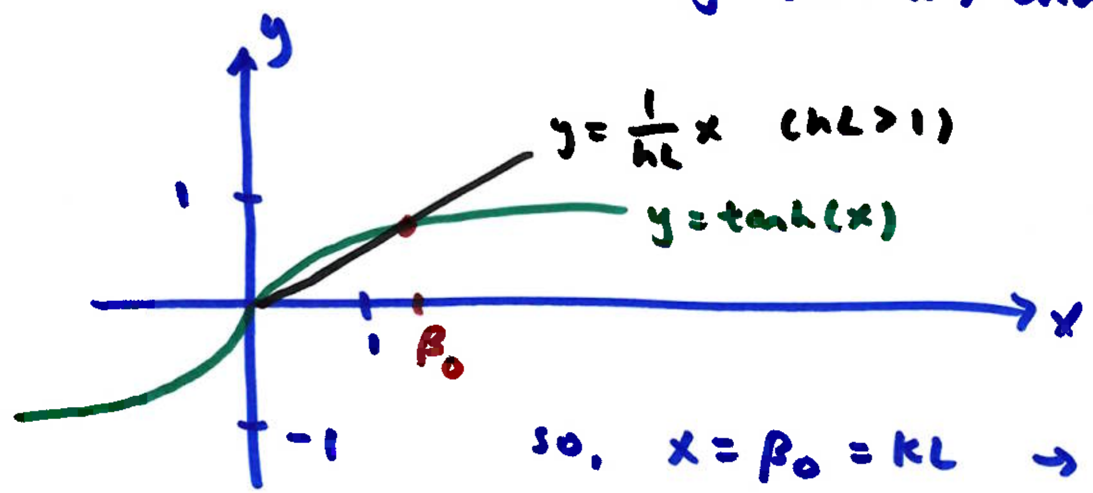
solve for K

$h \sinh(kL) = k\cosh(kL)$

$\tanh(kL) = \frac{k}{h} = \frac{kL}{hL}$

↳ solve $\tanh(x) = \frac{x}{hL}$ $\quad x = kL$

intersection of $y = \tanh(x)$ and $y = \frac{1}{hL}x$



so, $x = \beta_0 = kL \to k = \frac{\beta_0}{L}$

so $\boxed{\begin{aligned} \lambda &= -k^2 = -\frac{\beta_0^2}{L^2} \\ y &= \sinh\left(\frac{\beta_0}{L}x\right) \end{aligned}}$

eigenvalue, eigenfunction for $\lambda < 0$

next, $\lambda = 0$, then $\lambda > 0$