10.1 Sturm-Liouville problem (part 2)

continue from last time

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0$

$hy(L) - y'(L) = 0 \qquad h > 0$

we determined that λ can be negative

we found, for $\lambda < 0$, $\boxed{\lambda_0 = -\frac{\beta_0^2}{L^2}}$ $\boxed{y_0 = \sinh\left(\frac{\beta_0}{L}x\right)}$

β_0 is solution to $\tanh(x) = \frac{1}{hL}x$

now for $\lambda = 0$

$y'' + \lambda y = 0 \rightarrow y'' = 0 \quad$ so $\quad y = c_1 x + c_2$

$y(0) = 0 \rightarrow 0 = c_2$, so $y = c_1 x$ and $y' = c_1$

$hy(L) - y'(L) = 0 \rightarrow hc_1 L - c_1 = 0 \rightarrow c_1(hL - 1) = 0 \qquad c_1 \neq 0$ (nontrivial only)

so, $hL = 1 \qquad \boxed{\lambda_1 = 0} \qquad \boxed{y_1 = x}$

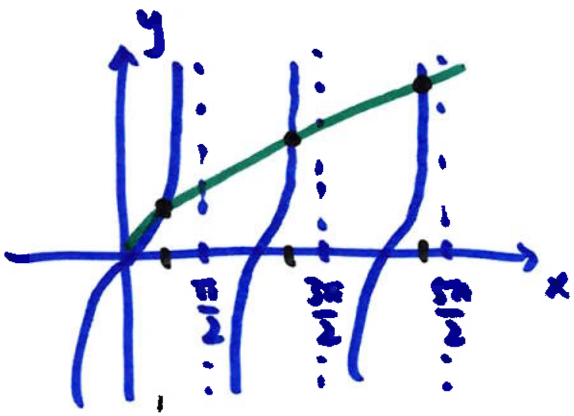
now for $\lambda > 0$

let $\lambda = K^2$

$y'' + \lambda y = 0 \rightarrow y'' + K^2 y = 0 \rightarrow y = C_1 \cos(Kx) + C_2 \sin(Kx)$

$y(0) = 0 \rightarrow 0 = C_1$, so $y = C_2 \sin(Kx)$ and $y' = C_2 K \cos(Kx)$

$hy(L) - y'(L) = 0 \rightarrow 0 = hC_2 \sin(KL) - C_2 K \cos(KL)$ $\quad C_2 \neq 0$



$0 = h \sin(KL) - K \cos(KL)$

$\tan(KL) = \frac{K}{h} = \frac{KL}{hL}$

solve $\tan(x) = \frac{1}{hL} x$ where $x = KL$

$hL > 0$

intersections $\rightarrow \beta_n = K_n L = \sqrt{\lambda_n}\, L$

$$\boxed{\lambda_n = \frac{\beta_n^2}{L^2}}$$

$$\boxed{y_n = \sin\left(\frac{\beta_n}{L} x\right)} \qquad n = 1, 2, 3, \ldots$$

solution to the problem is linear combo of all eigenfunctions fundamental solutions / eigenfunctions solved this way are all mutually orthogonal

$a<x<b$ $$\int_a^b y_i(x)\, y_j(x)\, dx = 0 \quad \text{if } i \neq j$$

for example, $y'' + \lambda y = 0 \qquad 0<x<\pi$

$y(0) = y(\pi) = 0$

gives us $\lambda_n = n^2$ and $y_n = \sin(nx)$ $n = 1, 2, 3, \ldots$

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \frac{\sin\overbrace{(m-n)}^{\text{integer}}x}{2(m-n)} - \frac{\sin\overbrace{(m+n)}^{\text{integer}}x}{2(m+n)}\Bigg|_0^\pi \quad (m \neq n)$$

$$= \frac{\sin(\text{integer}\ \pi)}{\ldots} - \frac{\sin(\text{integer}\ \pi)}{\ldots} = 0$$

if $n = m$

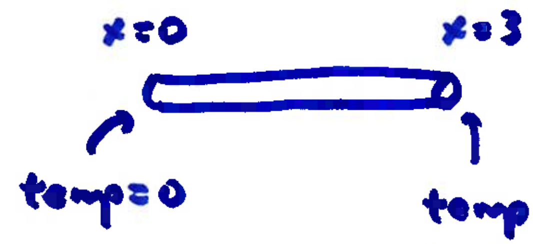
$$\int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \frac{1}{2}(1 - \cos(2nx))\,dx = \frac{\pi}{2}$$

we can expand functions using these orthogonal eigenfunctions (Fourier series is one particular example w/ $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$ as eigenfunctions)

<u>example</u> $y'' + \lambda y = 0 \qquad 0 < x < 3$

$y(0) = 0$

$y(3) + y'(3) = 0$



temp here is such that $y(3) = -y'(3)$ (function of heat flow through $x=3$)

this is a regular SL problem w/ $\lambda_n = \frac{\beta_n^2}{9}$ where β_n are solutions to $\tan(x) = -\frac{1}{9}x$

$(x = 2.8, 5.7, 8.7, 11.7, \ldots)$

and $y_n = \sin\left(\frac{\beta_n x}{3}\right)$

let's expand $f(x)=1$ using these y_n's

$$1 = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + \cdots = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n x}{3}\right)$$

since $\int_0^3 y_i y_j \, dx = 0$ if $i \neq j$

if both sides are multiplied by, for example, y_1

$$y_1 = c_1 y_1 y_1 + c_2 y_2 y_1 + c_3 y_3 y_1 + \cdots$$

$$\int_0^3 y_1 \, dx = \underbrace{\int_0^3 c_1 y_1 y_1 \, dx}_{\neq 0} + \underbrace{\int_0^3 c_2 y_2 y_1 \, dx}_{0} + \underbrace{\int_0^3 c_3 y_3 y_1 \, dx}_{0} + \underbrace{\cdots}_{0}$$

generalized

$$\int_0^3 1 \cdot y_n \, dx = \int_0^3 c_n (y_n)^2 \, dx$$

$$c_n = \frac{\int_0^3 1 \cdot y_n \, dx}{\int_0^3 (y_n)^2 \, dx} = \frac{\int_0^3 \sin\left(\frac{\beta_n x}{3}\right) dx}{\int_0^3 \sin^2\left(\frac{\beta_n x}{3}\right) dx} = \boxed{\frac{2 - 2\cos(\beta_n)}{\beta_n - \sin(\beta_n)\cos(\beta_n)}}$$