6.1 Stability and the Phase Planes

autonomous system : $\frac{dx}{dt} = f(x,y)$

$\frac{dy}{dt} = g(x,y)$

no time "t" in here

these can be linear or nonlinear, homogeneous or nonhomogeneous

critical points : (x,y) where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

<u>example</u>

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x-y \\ \frac{dy}{dt} &= x-3y \end{aligned}\right\} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\underbrace{\vec{x}\,' = A\vec{x}}_{}$$

possible only if linear system

can be nonhomogeneous

critical pts: $\frac{dx}{dt} = 0 \to 2x - y = 0$

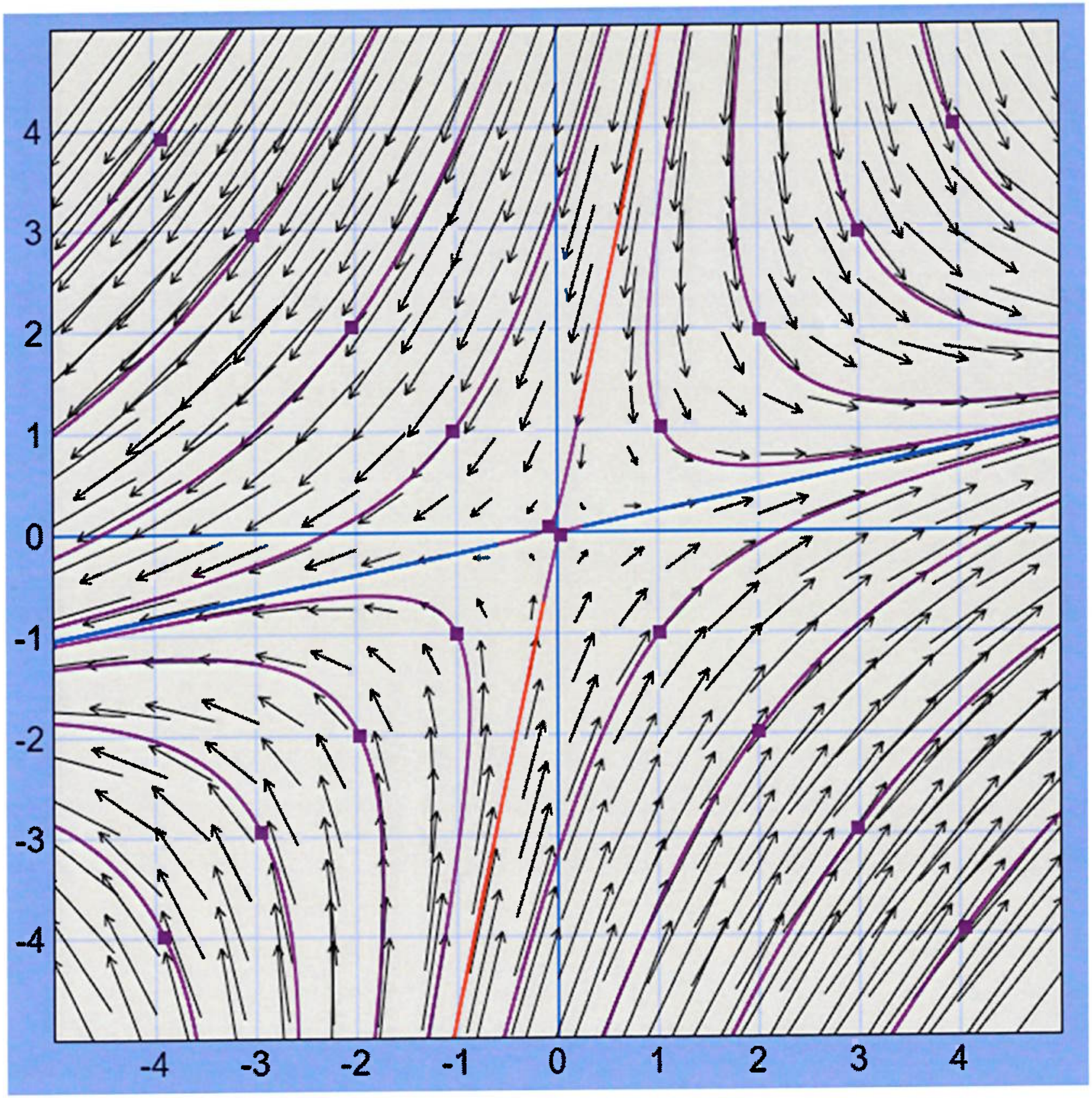
$\frac{dy}{dt} = 0 \to x - 3y = 0 \to x = 3y$

$2(3y) - y = 0 \quad 5y = 0 \to y = 0 \quad x = 0$

$\boxed{\text{cp: } (0,0)}$

ALL linear homogeneous systems $\vec{x}' = A\vec{x}$ has $(0,0)$ as cp.

Phase diagram



(0,0) is a saddle point

and is unstable

can stay at (0,0) only if started there

<u>example</u>

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - y + 1 \\ \frac{dy}{dt} &= x - 3y - 2 \end{aligned}\right\} \vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{x}' = A\vec{x} + \vec{u}$$

so nonhomogeneous

cp: $x' = 0 \rightarrow 2x - y + 1 = 0 \quad - ①$

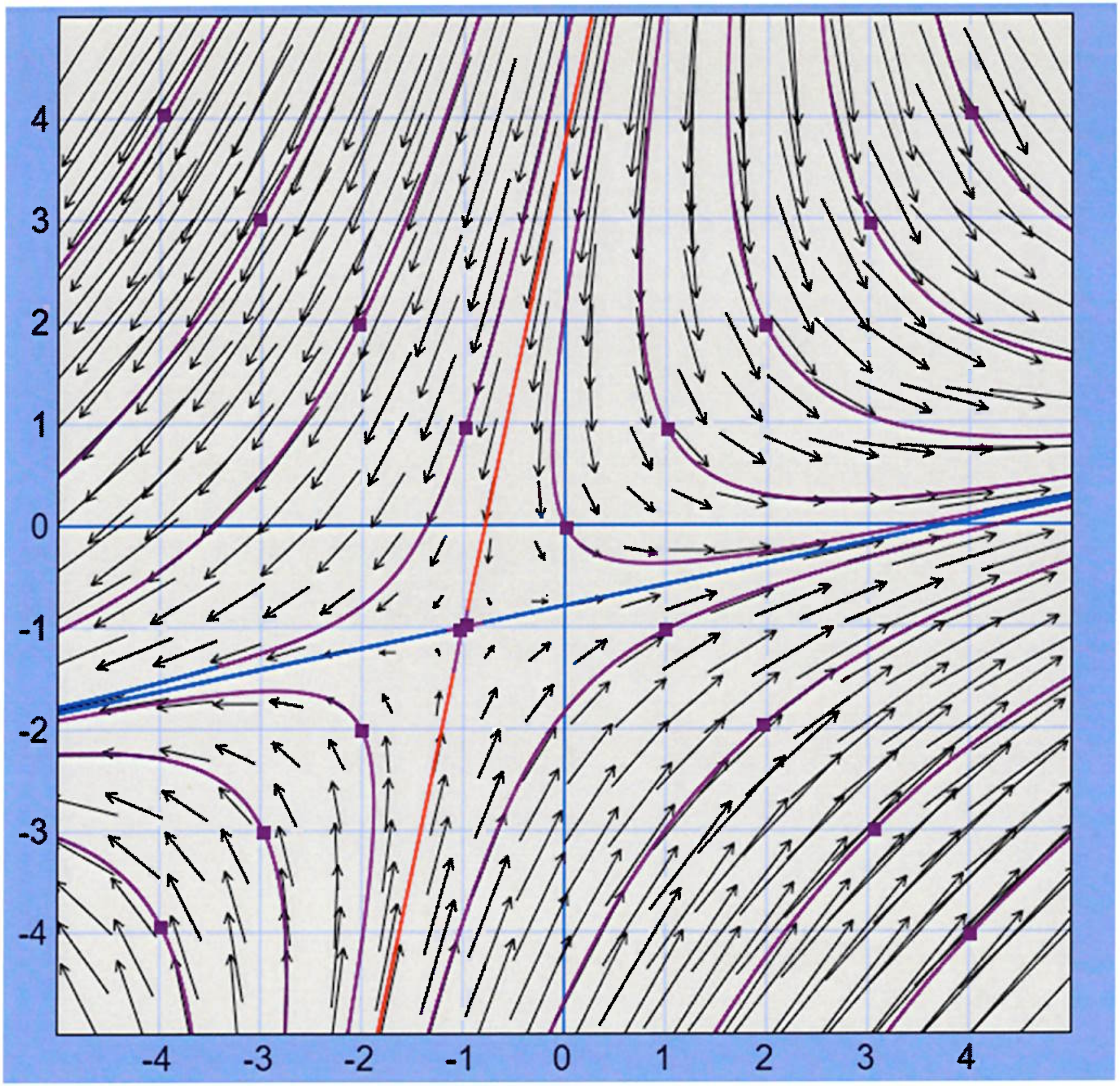
$y' = 0 \rightarrow x - 3y - 2 = 0 \quad - ②$

$2x - 6y - 4 = 0 \quad - ③$

$③ - ① \quad -5y - 5 = 0 \rightarrow y = -1 \quad x = -1$

$\boxed{cp: (-1, -1)}$

phase diagram



cp : (-1, -1)

saddle pt

unstable

$\vec{x}' = A\vec{x} + \vec{u}$

Same solutions

shifted to

other cp

(as $\vec{x}' = A\vec{x}$)

example

$$\frac{dx}{dt} = 1 - y^2$$ nonlinear (y^2)

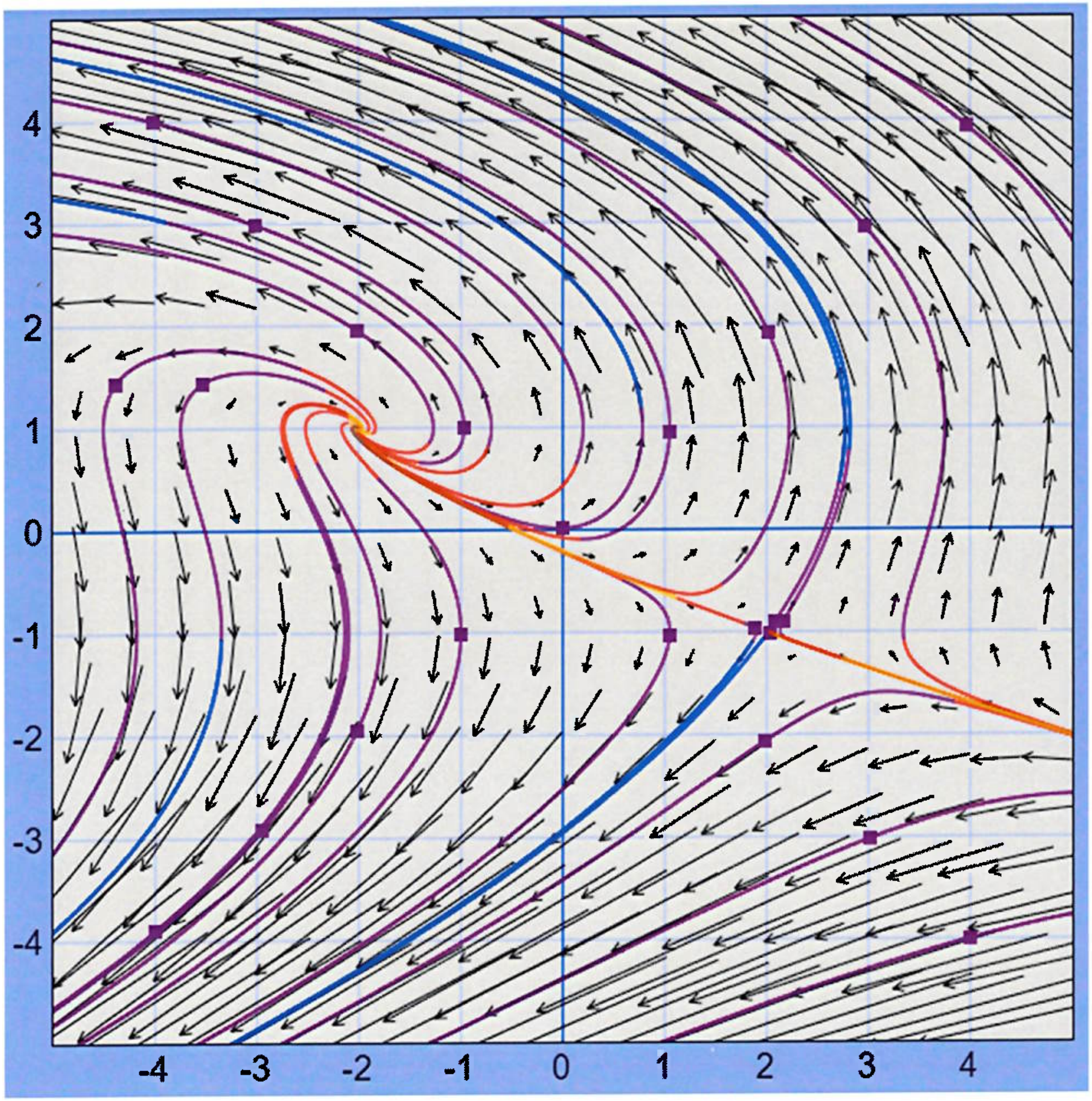
nonhomogeneous (1)

$$\frac{dy}{dt} = x + 2y$$ coupled (x, y are on the right together)

cp: $1 - y^2 = 0 \rightarrow y = 1, -1$

$x + 2y = 0 \rightarrow x = -2, 2$

$\boxed{(-2, 1), (2, -1)}$



$(-2,1)$

Spiral source

unstable

(any small perturbation from $(-2,1)$ goes away from $(-2,1)$ as $t\to\infty$)

$(2,-1)$

unstable saddle pt

example $\frac{dx}{dt} = x$ linear (no xy terms, no x^n or y^n $n \neq 1$, $n \neq 0$)

$\frac{dy}{dt} = -y$ homogeneous (no non x or y terms)

solvable : $\frac{dx}{dt} = x \rightarrow \frac{1}{x}dx = dt$ separable

$\frac{dy}{dt} = -y$

$$\int \frac{1}{x} dx = \int dt$$

$$\ln|x| = t + C$$

$$x = e^{t+C} = e^t e^C$$

$$\boxed{x = C_1 e^t}$$

Similarly, $\boxed{y = C_2 e^{-t}}$

phase diagram? $\rightarrow$ can we find y as function of x?

$\frac{dx}{dt} = x \quad \frac{dy}{dt} = -y$

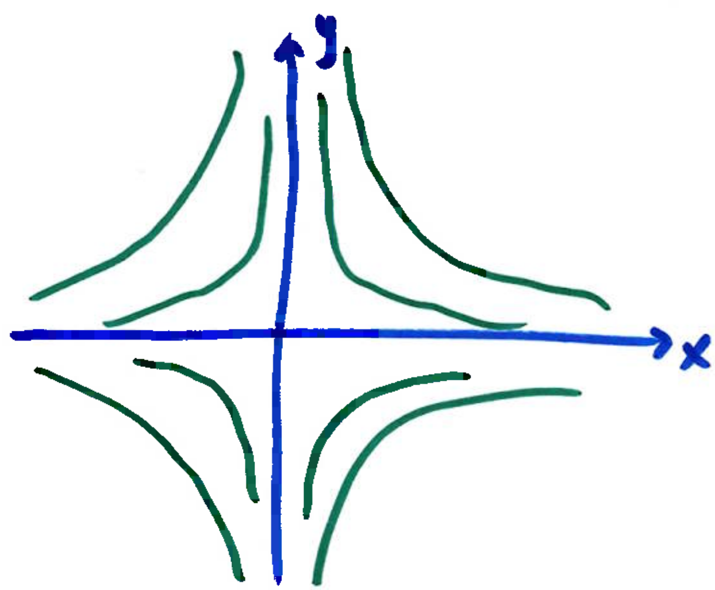
$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{-y}{x} \rightarrow \frac{1}{y}\,dy = -\frac{1}{x}\,dx$

integrate: $\ln y = -\ln x + c$

$y = e^{\ln x^{-1} + c} = e^{\ln x^{-1}} \cdot e^{c}$

$y = C x^{-1}$

$y = C \cdot \frac{1}{x}$ shape of solutions : hyperbolas



cp: $(0, 0)$

saddle pt

unstable

example $x'' + 4x - x^3 = 0$

2nd order → 2 first order eqs.

define $y = x'$

then, of course, $x' = y$ → 1st eq.

$y' = x'' = x^3 - 4x$ from given ODE

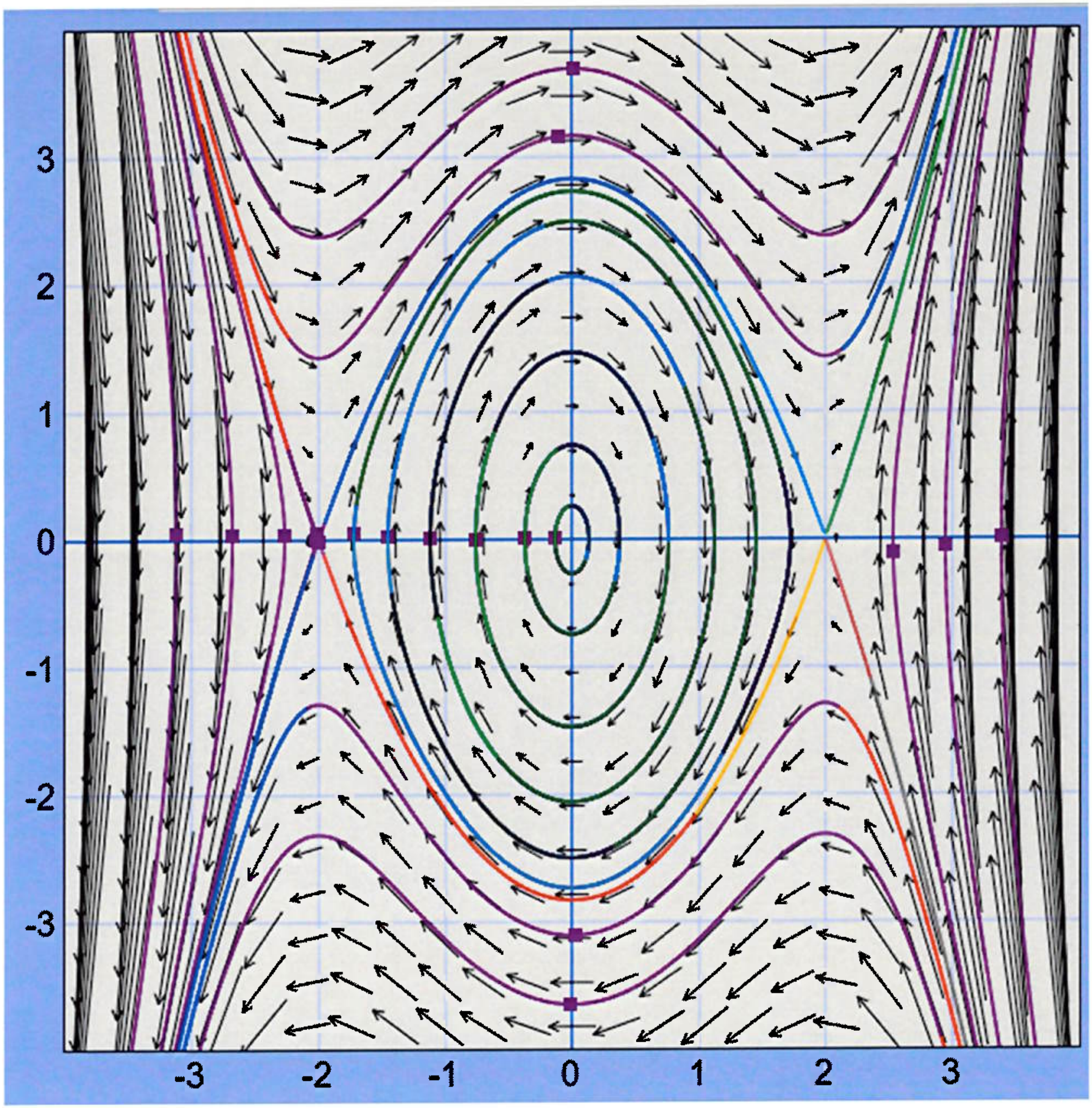
sys: $\boxed{\begin{aligned} x' &= y \\ y' &= x^3 - 4x \end{aligned}}$ nonlinear homogeneous

cp: $y = 0$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0 \rightarrow 0, 2, -2$

$\boxed{\text{cp: } (0,0), (2,0), (-2,0)}$



(0,0)
center
Stable