

6.2 Linear and Almost Linear Systems (part 2)

last time: the phase diagram of associated linear system resembles that of the nonlinear system near a critical pt unless if the associated linear system has:

- 1) repeated eigenvalues
- 2) purely imaginary eigenvalues

example

$$\frac{dx}{dt} = -3y + ay(x^2 + y^2) = f \quad a: \text{some constant}$$

$$\frac{dy}{dt} = 3x + ay(x^2 + y^2) = g$$

by inspection, one of the cp's is $(0, 0)$

linearize about $(0, 0)$

$$\vec{x}' = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} \vec{x}$$

$$\begin{aligned} f_x &= 2axy & f_y &= -3 + 3ay^2 + 2axy \\ g_x &= 3 + 2axy & g_y &= 3ay^2 \end{aligned}$$

$$= \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \vec{x} \quad \lambda = \pm 3i$$

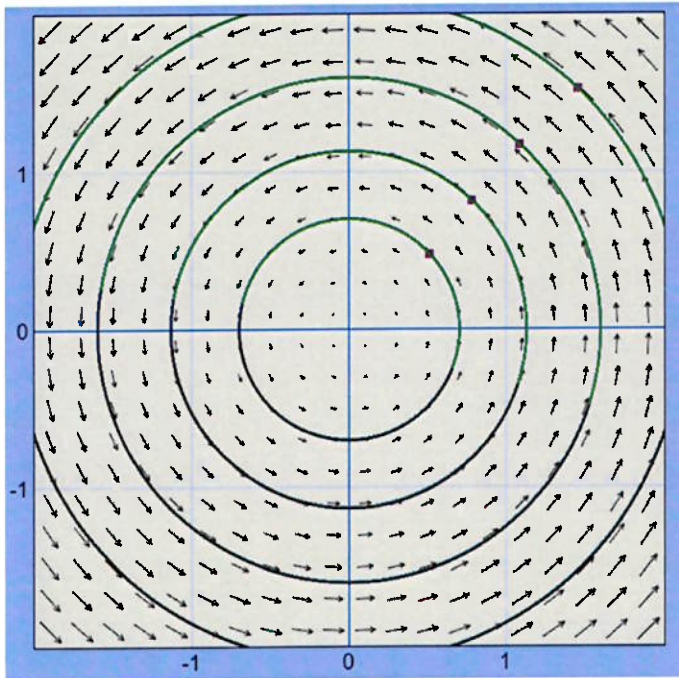
So linearized system is suggesting $(0,0)$ is a center (stable)
but the true behavior can be different.

$$\vec{x}' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \vec{x} + a \begin{bmatrix} y(x^2+y^2) \\ y(x^2+y^2) \end{bmatrix}$$

let's see the phase diagrams of cases when $a=1$, and $a=-1$
($a=0 \rightarrow$ center)

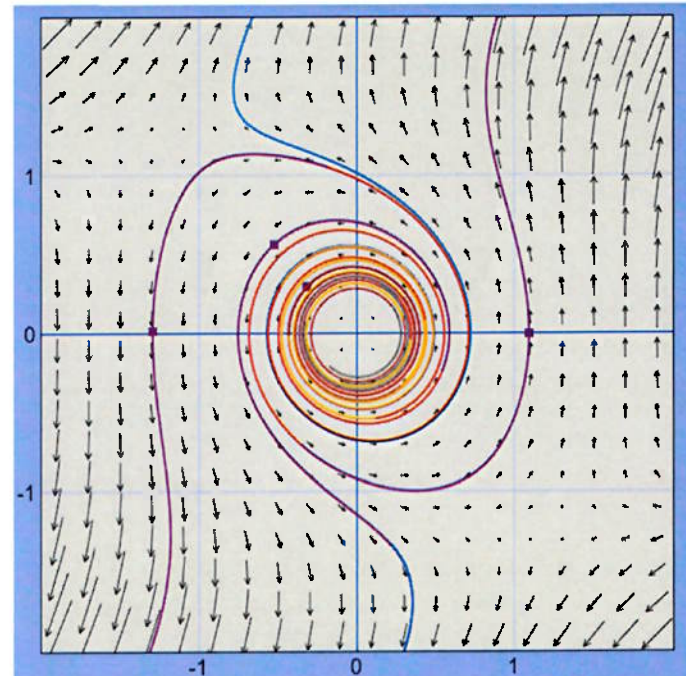
in general, nonlinear systems are hard to predict

case here: $\lambda = \text{purely imaginary}$
similar thing if $\lambda = \text{repeated}$ } "weak" behavior

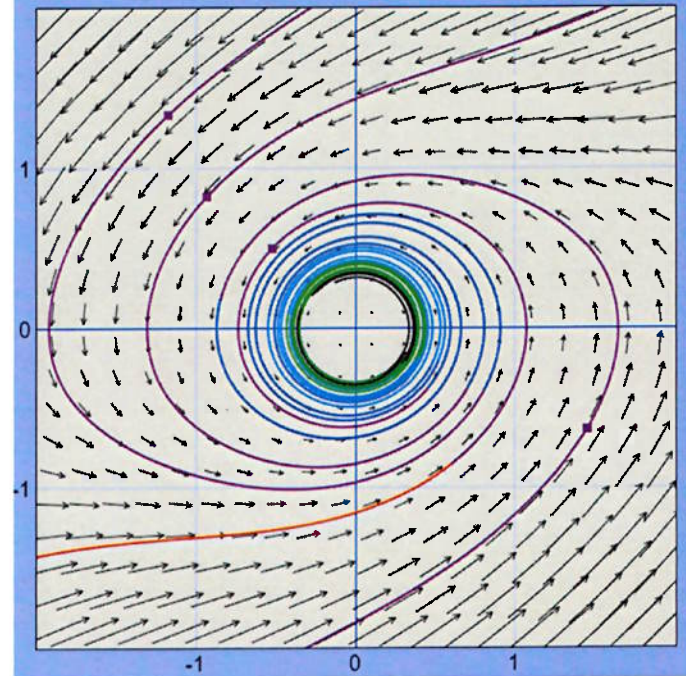


Linearized
(stable center)

Nonlinear $a=1$
(unstable
spiral point)



Nonlinear $a=-1$
(asymptotically
stable spiral
point)



$$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x} \quad \text{linear system}$$

eigenvalues: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$ $(a-\lambda)(d-\lambda) - bc = 0$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

clearly, a, b, c, d all can effect λ
and therefore type and stability of cp

can small perturbations to the numbers affect behavior?
are there ~~any~~ bifurcating behaviors?

example $\frac{dx}{dt} = -x + \epsilon y$ ← epsilon (small number)

$$\frac{dy}{dt} = x - y$$

$$\vec{x}' = \begin{bmatrix} -1 & \epsilon \\ 1 & -1 \end{bmatrix} \vec{x}$$

(p: (0,0))

$$\begin{vmatrix} -1-\lambda & \epsilon \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1+\lambda)^2 - \epsilon = 0$$

$$(1+\lambda)^2 = \epsilon$$

$$1+\lambda = \sqrt{\epsilon} \quad \text{or} \quad 1+\lambda = -\sqrt{\epsilon}$$

$$\lambda = -1 \pm \sqrt{\epsilon}$$

if $\epsilon = 0$, $\lambda = -1, -1$

asymptotically stable

proper or improper node

↓
complete matrix

↳ defective matrix

if $\varepsilon > 0$ (small pos. number)

$$\lambda = -1 \pm \sqrt{\text{small pos.}} = -1 + \sqrt{\varepsilon}, -1 - \sqrt{\varepsilon}$$

both still negative, AS improper node

if $\varepsilon < 0$ (small neg. #)

$$\lambda = -1 \pm \sqrt{\varepsilon} = -1 \pm i\sqrt{|\varepsilon|} \quad \text{AS spiral}$$

example

$$\vec{x}' = \begin{bmatrix} 1+\varepsilon & -5 \\ 1 & -1 \end{bmatrix} \vec{x}$$

cp: (0,0)

$$\begin{vmatrix} 1+\varepsilon-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4(4-\varepsilon)}}{2}$$

$$\varepsilon = 0 : \quad \lambda = \frac{\sqrt{-16}}{2} = \pm 2i \quad \text{stable, center}$$

$$\varepsilon = \text{small pos. \#} : \lambda = \frac{\varepsilon \pm \sqrt{\varepsilon - 16}}{2} = \overset{\nearrow > 0}{\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon - 16}{2}}} \approx \frac{\varepsilon}{2} \pm 2i$$

unstable spiral

$$\varepsilon = \text{small neg. \#} : \lambda = \frac{\varepsilon \pm \sqrt{\varepsilon - 16}}{2} \approx \overset{\nearrow < 0}{\frac{\varepsilon}{2} \pm 2i}$$

asympt. stable spiral

(very slowly spiraling in)