

7.1 Laplace Transform

type of integral transform (another common one is the Fourier transform)

Laplace Transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

kernel of
integral
transform

example $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^{\infty} 1 \cdot e^{-st} dt$$

treat s as constant
for integration

$$= \lim_{a \rightarrow \infty} \int_0^a e^{-st} dt = \lim_{a \rightarrow \infty} \left. \frac{1}{-s} e^{-st} \right|_0^a$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{1}{s} \underbrace{e^{-sa}}_{\substack{\text{goes to } 0 \\ \text{if } s > 0}} + \frac{1}{s} e^0 \right) = \boxed{\frac{1}{s}, s > 0}$$

Example $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^{\infty} t \cdot e^{-st} dt$$

$$= \lim_{a \rightarrow \infty} \int_0^a t \cdot e^{-st} dt$$

by parts

$$u = t \quad dv = e^{-st} dt$$

$$du = dt \quad v = -\frac{1}{s} e^{-st}$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{t}{s} e^{-st} \Big|_{t=0}^{t=a} + \int_0^a \frac{1}{s} e^{-st} dt \right)$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{t}{s} e^{-st} \Big|_{t=0}^{t=a} - \frac{1}{s^2} e^{-st} \Big|_{t=0}^{t=a} \right)$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{a}{s} \underbrace{e^{-sa}}_{\substack{\rightarrow 0 \\ s > 0}} - \frac{1}{s^2} \underbrace{e^{-sa}}_{\substack{\rightarrow 0 \\ s > 0}} + \frac{1}{s^2} \right) = \boxed{\frac{1}{s^2}, \quad s > 0}$$

$$\text{so } \mathcal{L}\{t\} = \frac{1}{s^2}, \quad s > 0$$

$$\mathcal{L}\{c_1 + c_2 t\} \quad c_1, c_2 \text{ constants}$$

$$= \int_0^{\infty} (c_1 + c_2 t) e^{-st} dt$$

$$= \int_0^{\infty} c_1 \cdot e^{-st} dt + \int_0^{\infty} c_2 t \cdot e^{-st} dt$$

$$= c_1 \underbrace{\int_0^{\infty} 1 \cdot e^{-st} dt}_{\mathcal{L}\{1\}} + c_2 \underbrace{\int_0^{\infty} t \cdot e^{-st} dt}_{\mathcal{L}\{t\}}$$

$$\text{so, } \mathcal{L}\{c_1 + c_2 t\} = c_1 \mathcal{L}\{1\} + c_2 \mathcal{L}\{t\} \quad \text{LT is linear}$$

$$\mathcal{L}\{c_1 \cdot f(t) + c_2 \cdot g(t)\}$$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

$$= c_1 F(s) + c_2 G(s)$$

provided there exist

$$\text{BUT } \mathcal{L}\{f(t) \cdot g(t)\}$$

$$\neq \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$$

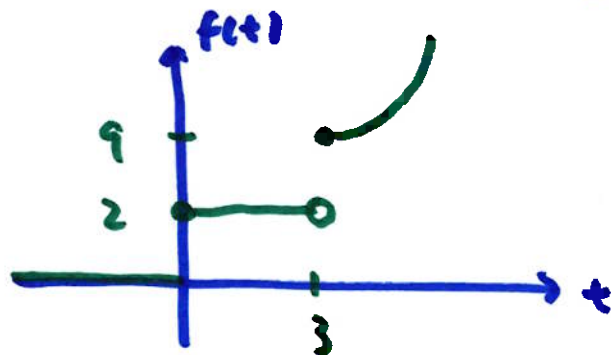
in general

as long as $f(t)$ is piecewise continuous, then $\mathcal{L}\{f(t)\}$ exists

↓
finite number of discontinuities

example

$$f(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \leq t < 3 \\ t^2 & t \geq 3 \end{cases}$$



$$F(s) = ?$$

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_0^3 2 \cdot e^{-st} dt + \int_3^{\infty} t^2 \cdot e^{-st} dt$$

$$= \dots = \frac{2 - 2e^{-3s}}{s} + \frac{e^{-3s}(3s+1)}{s^2} \quad (s > 0)$$

in practice, we don't usually do LT by hand

example $\mathcal{L}\{3t^4 - e^{5t} + 2\sin 3t\}$

$$= 3\mathcal{L}\{t^4\} - \mathcal{L}\{e^{5t}\} + 2\mathcal{L}\{\sin 3t\}$$

$$\underbrace{\mathcal{L}\{t^n\}} = \frac{n!}{s^{n+1}} \text{ from table}$$

$$= 3 \cdot \frac{4!}{s^5} - \frac{1}{s-5} + 2 \cdot \frac{3}{s^2+9}, \quad s > 5$$

inverse transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

use table from right to left

$$\mathcal{L}^{-1}\left\{\frac{2-4s}{s^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+16}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}$$

looks like

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2} \frac{4}{s^2+4^2}\right\} - 4\mathcal{L}^{-1}\{\dots\}$$

$$= \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} - \dots = \frac{1}{2} \sin 4t + \dots$$

$f(t)$	$F(s)$
1	$\frac{1}{s} \quad (s > 0)$
t	$\frac{1}{s^2} \quad (s > 0)$
$t^n \quad (n \geq 0)$	$\frac{n!}{s^{n+1}} \quad (s > 0)$
$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}} \quad (s > 0)$
e^{at}	$\frac{1}{s-a} \quad (s > a)$
$\cos kt$	$\frac{s}{s^2 + k^2} \quad (s > 0)$
$\sin kt$	$\frac{k}{s^2 + k^2} \quad (s > 0)$
$\cosh kt$	$\frac{s}{s^2 - k^2} \quad (s > k)$
$\sinh kt$	$\frac{k}{s^2 - k^2} \quad (s > k)$
$u(t-a)$	$\frac{e^{-as}}{s} \quad (s > 0)$