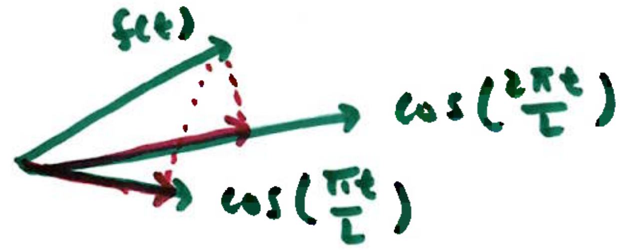
Fourier Series $f(t)$ period $2L$ for $-L<t<L$

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \underbrace{\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt}_{\text{projection onto } \cos\left(\frac{n\pi t}{L}\right)} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$



shifted version $0<t<2L$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

another version $f(t)$ period P. $0<t<P$

$$a_n = \frac{1}{P/2}\int_0^{P} f(t)\cos\left(\frac{n\pi t}{P/2}\right)dt \qquad b_n = \frac{2}{P}\int_0^{P} f(t)\sin\left(\frac{2n\pi t}{P}\right)dt$$

$$= \frac{2}{P}\int_0^{P} f(t)\cos\left(\frac{2n\pi t}{P}\right)dt$$

Fourier breaks up $f(t)$ into components explained by $\cos\left(\frac{n\pi t}{L}\right)$ and $\sin\left(\frac{n\pi t}{L}\right)$

Taylor breaks up $f(t)$ into components explained by t^n

$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$ converges to e^t

coefficients: $\left\{ 1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots, \frac{1}{n!}, \ldots \right\}$ sequence of coefficients

biggest coefficients $\rightarrow$ e^t is mostly 1 and t

Fourier does the same thing

$f(t) = t$ period 2π, $\quad 0 < t < 2\pi$

$a_0 = 2\pi$, $a_n = 0$, some b_n's but all $|b_n| < a_0$

mostly this, some sines, notably b_1, b_2,

biggest b : $b_1 = -2$ most dominant frequency $\frac{n\pi t}{L} = \frac{\pi t}{\pi} = 1t$

if this were a sound, that is the one most easily heard

Fourier series works with integer n, can't usually get $\cos(\pi t)$, for example

but we can use it to approximate $f(t) = 3\cos(2.01t)$

period 2π $\quad 0 < t < 2\pi$

a_2, b_2 by far more dominant since $\cos(2t)$ and $\sin(2t)$ match the frequency of $\cos(2.01t)$ the best among $\cos(nt)$ and $\sin(nt)$

Complex form of Fourier Series (we won't use in this class)

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

plug into $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right]$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

we get $f(t) \sim \sum_{n=-\infty}^{\infty} C_n e^{in\pi t/L}$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-in\pi t/L}\, dt \qquad n: -\infty \text{ to } \infty \text{ integers}$$

$$C_0 = \frac{a_0}{2} \qquad C_n = \frac{a_n - ib_n}{2} \qquad C_{-n} = \frac{a_n + ib_n}{2}$$

completely identical to the cosine/sine form

Fourier <u>Transform</u> : not integer n anymore but n as a real number

$f(t)$ doesn't have to be periodic

$(L \to \infty)$

Fourier Transform of $f(t)$ is

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

Laplace Transform of $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

in Laplace, s is complex : $s = \lambda + i\omega t$

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

Fourier $\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

$\downarrow$

cosine, sine

$$= \int_{-\infty}^{\infty} \left[f(t)\, e^{-\lambda t} \right] e^{-i\omega t}\, dt$$

Laplace is the Fourier transform of $f(t)\, e^{-\lambda t}$

$\downarrow$

breaks $f(t)$ into components explained by

$e^{\lambda t}\cos\omega t$ and $e^{\lambda t}\sin\omega t$

exponential sinusoids

LT is useful for ODEs because $e^{\lambda t}\cos\omega t$ or $e^{\lambda t}\sin\omega t$ show up (e.g. mass-spring-damper)