1 Cohomology of Affine Schemes

Thm Let R be a noetherian ring, $X = \operatorname{Spec} R$ and M an R-module. Then

$$H^i(X, \widetilde{M}) = 0 \text{ for } i > 0$$

First we need

Prop If I is an injective R-module Then $\widetilde{I}$ is a flasque sheaf.

[See Hartshorne III prop 3.4]

pf of thm

Given M, we can choose an injective resolution

$$0 \to M \to I^0 \to I^1 \to \cdots$$

By the prop, we have a flasque resolution

$$0 \to \widetilde{M} \to \widetilde{I}^0 \to \widetilde{I}^1 \to \cdots$$

Therefore $H^i(X, \widetilde{M}) = \dfrac{\operatorname{Ker} \Gamma(\widetilde{I}^i) \to \Gamma(\widetilde{I}^{i+1})}{\operatorname{im} \Gamma(\widetilde{I}^{i-1}) \to \Gamma(\widetilde{I}^i)}$

However,

$$\begin{array}{ccccc} \Gamma(\widehat{\mathcal{I}}^0) & \to & \Gamma(\widehat{\mathcal{I}}^1) & \to & \cdots \\ \shortparallel & & \shortparallel & & \\ \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \cdots \end{array}$$

Thus $H^i(X, \widehat{\mathcal{M}}) = 0$ for $i > 0$. //

2 Mayer-Vietoris

Thm Given an open cover $\{U, V\}$ of X and a sheaf $\mathcal{F}$, there exists a long exact seq.

$$\to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \cdots$$

where the middle two maps are the sum and differences of restrictions

3 Cohomology of $\mathbb{P}^1$

Let k be a field.

For the first application, we compute the cohomology $H^{\cdot}(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(n))$.

Let's start with $n=0$, i.e. $\mathcal{O}_{\mathbb{P}^1}$.

We use the standard cover

$$\mathcal{U}_0 = \operatorname{Spec} k\left[\frac{x_1}{x_0}\right] = \operatorname{Spec} k[t]$$

$$\mathcal{U}_1 = \operatorname{Spec} k\left[\frac{x_0}{x_1}\right] = \operatorname{Spec} k[t^{-1}]$$

$$\mathcal{U}_{01} := \mathcal{U}_0 \cap \mathcal{U}_1 = \operatorname{Spec} k[t, t^{-1}]$$

Then Mayer-Vietoris gives

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}) \to H^0(\mathcal{U}_0, \mathcal{O}) \oplus H^0(\mathcal{U}_1, \mathcal{O}) \to H^0(\mathcal{U}_{01}, \mathcal{O}) \to H^1(\mathbb{P}^1, \mathcal{O}) \to H^1(\mathcal{U}_0, \mathcal{O}) \oplus H^1(\mathcal{U}_1, \mathcal{O})$$

This can be identified with

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}) \to k[t] \oplus k[t^{-1}] \xrightarrow{\delta} k[t, t^{-1}] \to H^1(\mathbb{P}^1, \mathcal{O}) \to 0$$

Continuing the sequence, we see
$$H^i(\mathbb{P}^1, \mathcal{O}) = 0 \text{ when } i \geq 2$$
In fact, the same argument shows

Prop $H^i(\mathbb{P}^1, \mathcal{F}) = 0$ for $i \geq 2$ and $\mathcal{F}$ quasicoherent.

To analyze the lower indices we note
$$\delta(f(t), g(t^{-1})) = f(t) - g(t^{-1})$$
So that
$$H^0(\mathbb{P}^1, \mathcal{O}) = \ker \delta = k[t] \cap k[t^{-1}] = k$$

& $H^1(\mathbb{P}^1, \mathcal{O}) = \operatorname{coker} \delta = 0$ since δ is clearly surjective.

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In general, recall

$$\check{H}^1(\{U_0, U_1\}, \mathcal{O}^*) = \frac{\{g_{01} \in k[t,t^{-1}]^*\}}{\{f_0/f_1 \mid f_0 \in k[t]^*,\ f_1 \in k[t^{-1}]^*\}}$$

Since $k[t,t^{-1}]^* = \{a t^n \mid a \in k^*, n \in \mathbb{Z}\}$

& $k[t^{\pm}]^* = k^*$

We see that

$$\check{H}^1(\{U_i\}, \mathcal{O}^*) \cong \mathbb{Z}$$

Under this isomorphism $n \in \mathbb{Z}$ to the class of $g_{01} = t^n$

Recall $\check{H}^1(X, \mathcal{O}^*) = \mathrm{Pic}(X) =$ group of line bundles

for any (nice) scheme X

The line bundle corresponding to $g_{01} = t^n$ is precisely $L = \mathcal{O}_{\mathbb{P}^1}(n)$.

Recall that to go from L to the cocycle, we use a diagram

$$\begin{tikzcd} L|_{U_0} & \mathcal{O}_{U_0} \arrow[l, "\sim"', "\varphi_0"] \arrow[d, dashed, "g_{01}"] \\ L|_{U_1} \arrow[r, "\sim", "\varphi_1"'] & \mathcal{O}_{U_1} \end{tikzcd}$$

Let us view φ_0 as the reference iso. Then $1 \in \mathcal{O}_{U_0}$ goes to $t^n \in \mathcal{O}_{U_1}$ under g_{01}. Then M.V can be identified with

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}(n)) \to k[t] \oplus k[t^{-1}] \xrightarrow{\delta} k[t, t^{-1}] \to H^1(\mathbb{P}^1, \mathcal{O}(n)) \to 0$$

wh

$$\delta(f(t), g(t^{-1})) = f(t) - t^n g(t^{-1})$$

So

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \operatorname{Ker} \delta$$
$$= k[t] \cap t^n k[t^{-1}]$$

when $n < 0$, we get

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = 0$$

when $n \geq 0$

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \langle 1, t, \dots t^n \rangle$$

$$\cong k^{n+1}$$

For

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) = \operatorname{coker} \delta$$

$$= \frac{k[t, t^{-1}]}{k[t] + t^n k[t^{-1}]}$$

$$= \frac{k[t, t^{-1}]}{\langle \dots t^{n-1}, t^n, 1, t, \dots \rangle}$$

When $n \geq -1$

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$$

If $n \leq -2$,

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) \cong k^{-n+2}$$

Summarizing

Thm

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) \cong \begin{cases} k^{n+1} & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) \cong \begin{cases} k^{-n+2} & \text{if } n \leq -2 \\ 0 & \text{otherwise} \end{cases}$$

Thm If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^1_k$, then

$H^i(\mathbb{P}^1, \mathcal{F})$ is finite dimensional

$\forall i$

pf We saw $H^i(\mathbb{P}^1, \mathcal{F}) = 0$ when $i \geq 2$. So we just have to prove finiteness for $i = 0, 1$.

Recall $\mathcal{F} = \widetilde{M}$ where M is a f.g. graded $S = k[x_0, x_1]$ module.

We can find an exact seq. of modules

$$0 \to K \to \bigoplus_{i=1}^{N} S(n_i) \to M \to 0$$

giving an exact seq of coherent sheaves

$$0 \to \underset{\substack{\shortparallel \\ \mathcal{K}}}{\widetilde{K}} \to \bigoplus \mathcal{O}(n_i) \to \mathcal{F} \to 0$$

We obtain

$$\bigoplus H^1(\mathcal{O}(n_i)) \to H^1(\mathcal{F}) \to 0$$

which implies finiteness for H^1.

This applies to $\mathcal{K}$. Then

$$\bigoplus H^0(\mathcal{O}(n_i)) \to H^0(\mathcal{F}) \to H^1(\mathcal{K})$$

gives finiteness of $H^0(\mathcal{F})$. //

4 Čech Cohomology

We want to extend the Mayer-Vietoris technique to several open sets. This leads to (higher) Čech cohomology.

Given an open cover $\mathcal{U} = \{U_i\}$ of a space X, set

$$U_{i_1 \dots i_n} = U_{i_1} \cap \dots \cap U_{i_n}$$

Given a shf $\mathcal{F} \in Ab(X)$, a Čech p-cochain is a collection

$$\alpha_{i_0 \dots i_p} \in \mathcal{F}(U_{i_0, \dots i_p})$$

s.t. $\alpha_{i_0 \dots i_p} = 0$ if indices are repeated

and $\alpha_{\sigma(i_0) \dots \sigma(i_p)} = \operatorname{sign}(\sigma)\, \alpha_{i_0 \dots i_p}$

for any permutation σ.

Let $\check{C}^p(\mathcal{U}, \mathcal{F})$ denote group of p-cochains

We define the Čech coboundary by

$$\partial : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

by

$$(\partial \alpha)_{i_0 \ldots i_{p+1}} = \sum (-1)^k \alpha_{i_0 \ldots \hat{i_k} \ldots i_{p+1}} \Big|_{U_{i_0 \ldots i_p}}$$

(Here $\hat{}$ means omit.)

Lemma $\partial^2 = 0$

Def The pth Čech cohomology group

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(C^\bullet(\mathcal{U}, \mathcal{F}))$$

$$= \frac{\ker \partial : C^p \to C^{p+1}}{\operatorname{im} \partial\, C^{p-1} \to C^p}$$

One can take the limit

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(X, \mathcal{F})$$

as before. However, there example, whe

$$\boxed{\check{H}^p(X, \mathcal{F}) \not\cong H^p(X, \mathcal{F})}$$

The conclusion is $\check{H}^p$ is not good for theoretical purposes, but it is good for computation. Here is the key def/result.

Def An open cover $\mathcal{U}$ is called a Leray cover w.r.t to $\mathcal{F}$ if

$$H^p(\mathcal{U}_{i_1 \dots i_q}, \mathcal{F}) = 0 \quad \forall p > 0$$
$$\forall i_1 \dots i_q$$

Thm If $\mathcal{U}$ is Leray w.r.t to $\mathcal{F}$ then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F}) \quad \forall p.$$

We give the proof after some lemmas.

lemma For any open cover $\mathcal{U}$

$$H^0(X, \mathcal{F}) = \check{H}^0(\mathcal{U}, \mathcal{F})$$

pf There is a homomorphism

$$H^0(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
$$f \longmapsto (f|_{U_i})_i$$

This is an isomorphism because $\mathcal{F}$ is a sheaf //

lemma If

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is exact in $Ab(X)$ and

$$H^1(U_{i_0 \ldots i_p}, \mathcal{A}) = 0 \quad \forall i_0 \ldots i_p$$

Then there is a long exact seq.

$$\to \check{H}^1(\mathcal{U}, \mathcal{A}) \to \check{H}^1(\mathcal{U}, \mathcal{B}) \to \cdots$$

pf The hypothesis implies there is a short exact seq

$$0 \to C^\cdot(\mathcal{U}, \mathcal{A}) \to C^\cdot(\mathcal{U}, \mathcal{B}) \to C^\cdot(\mathcal{U}, \mathcal{C}) \to 0$$

This gives a long exact seq. //

lemma If $\mathcal{F}$ is flasque, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ $\forall p > 0$.

pf of thm We know the $p=0$ case, so we need to check it for $p>0$.

Consider the seq.

$$0 \to \mathcal{F} \to \mathcal{G}(\mathcal{F}) \to \mathcal{C}(\mathcal{F}) \to 0$$

from long ago. We have

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{F}) \arrow[r] \arrow[d, equal] & H^0(\mathcal{G}(\mathcal{F})) \arrow[r] \arrow[d, "\wr"] & H^0(\mathcal{C}(\mathcal{F})) \arrow[r] \arrow[d, equal] & H^1(\mathcal{F}) \arrow[r] \arrow[d, "v"] & 0 \\
(*)\ 0 \arrow[r] & \check{H}^0(\mathcal{F}) \arrow[r] & \check{H}^0(\mathcal{G}(\mathcal{F})) \arrow[r] & \check{H}^0(\mathcal{C}(\mathcal{F})) \arrow[r] & \check{H}^1(\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

The arrow v is an iso by the 5-lemma. So the $p=1$ case is done.

From $(*)$, $\check{H}^1(\mathcal{C}(\mathcal{F})) \xrightarrow{\sim} \check{H}^2(\mathcal{F})$

Now we have

$$\begin{tikzcd}
H^2(\mathcal{F}) \arrow[r, phantom, "\cong"] \arrow[d] & H^1(\mathcal{C}(\mathcal{F})) \arrow[d, "\wr"] \\
\check{H}^2(\mathcal{F}) \arrow[r, phantom, "\cong"] & \check{H}^1(\mathcal{C}(\mathcal{F}))
\end{tikzcd}$$

This proves $p=2$, etc.

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§ Cohomology of $\mathbb{P}^n$

Thm Suppose X is a separated noetherian scheme and $\mathcal{F}$ is coherent (or even quasicoherent). Then

for any affine open cover $\mathcal{U}$ of X

$$H^p(X, \mathcal{F}) \cong \check{H}^p(\mathcal{U}, \mathcal{F})$$

Pf Since $U_{i_1 \dots i_q}$ is affine and noetherian (by the assumptions)

$$H^p(U_{i_1 \dots i_q}, \mathcal{F}) = 0 \quad \forall p > 0$$

Therefore $\mathcal{U}$ is Leray. //

Now we come to the key calculation

Thm Fix $\mathbb{P} = \mathbb{P}^d_R$, when R is noetherian.

Then

(1) $H^0(\mathbb{P}, \mathcal{O}(n)) \cong \underbrace{R[x_0, \ldots x_d]_n}_{\text{homog poly of deg } n \text{ in above variables}}$

(2) $H^i(\mathbb{P}, \mathcal{O}(n)) = 0$ for $i \neq 0, d$

(3) $H^d(\mathbb{P}, \mathcal{O}(n)) \cong H^0(\mathbb{P}, \mathcal{O}(-n-d-1))^*$

Rmk $R[x_0, \ldots x_d]_n$ is a free module of rank $\binom{n+d}{d}$

by well known counting formulas.