1 Cup Product

Given a ringed space (X, R) and sheaves of R-modules $\mathcal{F}, \mathcal{G}$, we define $\mathcal{F} \otimes_R \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{R(U)} \mathcal{G}(U)$$

We have a natural map

$$\cup : H^0(X, \mathcal{F}) \otimes_{R(X)} H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{G})$$

$$f \cup g = f \otimes g$$

Given Čech cochains $(f_{i_0 \ldots i_p}) \in C^p(\mathcal{U}, \mathcal{F})$ and $(g_{j_0, \ldots j_q}) \in C^q(\mathcal{U}, \mathcal{G})$

$$(f \cup g)_{i_0 \ldots i_{p+q}} = f_{i_0 \ldots i_p} \otimes g_{i_p \ldots i_{p+q}}$$

Lemma $\quad \partial (f \cup g) = \pm (\partial f) \cup g \pm f \cup (\partial g)$

Therefore we get a cup product

$$\cup : \check{H}^p(\mathcal{U}, \mathcal{F}) \otimes \check{H}^q(\mathcal{U}, \mathcal{G}) \to \check{H}^{p+q}(\mathcal{U}, \mathcal{F} \otimes \mathcal{G})$$

2 Cohomology of $\mathbb{P}^d$

Last time we stated

Thm Fix $\mathbb{P} = \mathbb{P}^d_R$, where R is noetherian.

Then

(a) $H^0(\mathbb{P}, \mathcal{O}(n)) \cong \underbrace{R[x_0, \dots x_d]_n}_{\substack{\text{homog poly of deg } n \\ \text{in above variables}}}$

(b) $H^i(\mathbb{P}, \mathcal{O}(n)) = 0$ for $i \neq 0, d$

(c) $H^d(\mathbb{P}, \mathcal{O}(n)) \cong H^0(\mathbb{P}, \mathcal{O}(-n-d-1))^*$

The main idea is use Čech cohology w.r.t the standard open cover $\mathcal{U}_i = D_+(x_i)$

The other trick is to consider all the $\mathcal{O}(n)$ at the same time by considering the graded R-module $\bigoplus H^i(\mathcal{O}(n))$.

Once (a) is proved, we will see that this is a graded S-module w.r.t. cup product where

Let $S = k[x_0, \dots x_n]$

The Čech complex is

$C^\cdot(\mathcal{U}, \oplus \mathcal{O}(n)) =$

$$\bigoplus_i S[\tfrac{1}{x_i}] \to \bigoplus_{i,j} S[\tfrac{1}{x_i x_j}] \to \dots S[\tfrac{1}{x_0 \cdots x_d}]$$

Since the complex has length d, we get

<u>lemma</u> $H^i(\oplus \mathcal{O}(n)) = 0$ for $i > d$

This is a special case of (b).

Next we prove (a).

<u>Prop</u> $H^0(\bigoplus \mathcal{O}(n)) \cong S$ as a graded R module.

<u>pf</u>. Let $T = S[\tfrac{1}{x_0 \cdots x_d}]$. $S[\tfrac{1}{x_i}]$ can be identified with subrings of T, and

One checks that any homogeneous element of T has a unique representation as

$$x_0^{i_1} \cdots x_d^{i_d} \, f(x_0, \dots x_d)$$

where no $x_j \nmid f$

One sees from this that

$$H^0(\bigoplus \mathcal{O}(n)) \cong \bigcap_i S[\tfrac{1}{x_i}]$$

$$= S \qquad //$$

Next we turn to (c)

Prop $H^d(\mathbb{P}, \mathcal{O}(n)) \cong S^*_{-n-d-1}$

pf: $$H^d(\bigoplus \mathcal{O}(n)) = \frac{S[\frac{1}{x_0 \cdots x_d}]}{\operatorname{im} \partial}$$

$$\cong \frac{\bigoplus_{i_j \in \mathbb{Z}} R\, x_0^{i_0} \cdots x_d^{i_d}}{\left(\sum_j S[\frac{1}{x_0 \cdots \hat{x}_j \cdots x_d}]\right)} \leftarrow \text{spanned by monomials with some nonneg. exponents.}$$

$$= \bigoplus_{i_j < 0} R\, x_0^{i_0} \cdots x_d^{i_d}$$

The grading is sum of exponents.

So $$H^d(\mathcal{O}(-d-1)) \cong R\, x_0^{-1}\cdots x_d^{-1} \cong R$$

The previous results gives a

$$H^0(\mathcal{O}(n)) = S_d = \bigoplus_{\sum i_k = d} R\, x_0^{i_0}\cdots x^{i_d}$$

The cup product

$$\cup : H^0(\mathcal{O}(n)) \times H^d(\mathcal{O}(-n-d-1)) \to H^d(\mathcal{O}(-d-1)) \cong R$$

which on the basis elements is given by

$$(x_0^{i_0}\cdots x^{i_d}) \cup (x_0^{j_0}\cdots x^{j_d}) = x_0^{i_0+j_0}\cdots x^{i_d+j_d}$$

This induces an isomorphism

$$H^d(\mathcal{O}(-n-d-1)) \cong H^0(\mathcal{O}(n))^*$$

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Finally we prove (5).

Prop If $0 < i < d$, then

$$H^i(\mathcal{O}_{\mathbb{P}^d}(n)) = 0$$

pf Setting $x_d = 0$ gives a hyperplane $H \cong \mathbb{P}^{d-1}_R$. We have an exact seq.

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{O}_H \to 0$$

Tensoring with $\mathcal{O}(n)$ gives

$$0 \to \mathcal{O}_{\mathbb{P}^d}(n-1) \to \mathcal{O}_{\mathbb{P}^d}(n) \to \mathcal{O}_{\mathbb{P}^{d-1}}(n) \to 0$$

By induction

$$H^i(\mathcal{O}_{\mathbb{P}^{d-1}}(n)) = 0, \quad 0 < i < d-1$$

We also have

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^d}(n)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(n)) \\ \| & & \| \\ S_d & \longrightarrow\!\!\!\!\rightarrow & R[x_0, \dots x_{d-1}] \end{array}$$

is surjective, and dually

$$\begin{array}{ccc} H^{d-1}(\mathcal{O}_{\mathbb{P}^{d-1}}(n)) & \longrightarrow & H^d(\mathcal{O}_{\mathbb{P}^d}(n)) \\ \| & & \| \\ S^*_{-n-d-1} & \longrightarrow & S^*_{-n-d-1} \end{array}$$

is injective.

Therefore

$$(*) \qquad H^i(\mathcal{O}_{\mathbb{P}^d}(n)) \xrightarrow{x_d} H^i(\mathcal{O}_{\mathbb{P}^d}(n))$$

is an isomorphism for $0 < i < d$

<u>Claim</u> Every element of the S-module

$$M = \bigoplus_n H^i(\mathcal{O}_{\mathbb{P}^d}(n)) \text{ is annihilated by}$$

a power of x_d for $i > 0$.

when combined with $(*)$, this implies the proposition.

The claim is equivalent to

$$(**) \quad M\left[\tfrac{1}{x_d}\right] = 0.$$

Since localization is exact, it is enough to show

$$H^i\left(C^\cdot(\mathcal{U}, \oplus \mathcal{O}(n))\left[\tfrac{1}{x_d}\right]\right) = 0 \text{ for } i > 0$$

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But

$$C^{\cdot}(\mathcal{U}, \bigoplus \mathcal{O}(n))\left[\frac{1}{x_d}\right] = C^{\cdot}(\{U_i \cap U_d\}, \bigoplus \mathcal{O}(n))$$

Since this computes the cohomology

$$H^{\cdot}(\mathcal{U}_d, \bigoplus \mathcal{O}(n))$$

We see this equals 0 for $i > 0$ //

3 Cohomology of $\mathbb{P}^d$ (cont)

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^d_R$, set

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^d}} \mathcal{O}_{\mathbb{P}^d}(n)$$

Def A coherent sheaf $\mathcal{F}$ on a scheme X is generated by global sections, or globally generated, if $\forall\, x \in X$ closed the natural map

$$H^0(X, \mathcal{F}) \longrightarrow \mathcal{F}_x$$

is surjective.

Thm Let R be a noetherian ring.
Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P} = \mathbb{P}^d_R$.
Then

a) $\forall\, i \geq 0$, $H^i(\mathbb{P}, \mathcal{F})$ is a finitely generated R-module, and $H^i(\mathbb{P}, \mathcal{F}) = 0$ when $i > d$.

b) $\exists\, n_0$, depending on $\mathcal{F}$, such that
(i) $H^i(\mathbb{P}, \mathcal{F}(n)) = 0 \quad \forall\, i > 0,\ n \geq n_0$
(ii) $\mathcal{F}(n)$ is generated by global sections for $n \geq n_0$.

Again we break the proof into a series of lemmas/props.

lemma 1 $H^i(\mathbb{P}, \mathcal{F}) = 0$ for $i > d$

pf The Čech complex w.r.t. the standard cover has length d. //

Lemma 2 There exists an exact seq. of coherent sheaves

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{N} \mathcal{O}(n_i) \to \mathcal{F} \to 0$$

pf Let $S = R[x_0, \dots x_d]$ as before.

Then $\mathcal{F} = \tilde{M}$, for some f.in gen graded S-module M.

Choose generators $m_i \in M_{n_i}$, $i=1,\dots N$.

We have a surjection

$$\bigoplus S(n_i) \xrightarrow{f} M$$

$$1 \in S(n_i) \longmapsto m_i$$

Let $K = \ker(f)$. Then we have an exact seq.

$$\begin{array}{ccccccc} 0 \to & \tilde{K} & \to & \bigoplus \widetilde{S(n_i)} & \to & \tilde{M} & \to 0 \\ \text{def} & \| & & \| & & \| & \\ & \mathcal{K} & & \bigoplus \mathcal{O}(n_i) & & \mathcal{F} & \end{array}$$

//

part of (a) and (b)(i) follows from

Prop 1

(1) $H^i(\mathbb{P}, \mathcal{F})$ is finitely generated

(2) if $n >> 0$, $H^i(\mathbb{P}, \mathcal{F}(n)) = 0$, $\forall\, i > 0$

pf We prove both parts by descending induction on i. When $i = >, d+1$, the prop follows from lemma 1.

In general, by lemma 2, we obtain

$$\bigoplus H^i(\mathcal{O}(n+n_i)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{K}(n))$$

it follows by induction and the previous thm that $H^i(\mathcal{F})$ is fin gen.

and that

$$H^i(\mathcal{F}(n)) = 0 \begin{cases} \text{for } n >> 0 \\ i > 0 \end{cases}$$

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Given a closed pt $x \in X$, let m_x be the ideal sheaf at x.

(with reduced structure).

We have an exact seq

$$0 \to \underbrace{m_x \cdot \mathcal{F}}_{\operatorname{im}(m_x \otimes \mathcal{F} \to \mathcal{F})} \longrightarrow \mathcal{F} \to \underbrace{\mathcal{F}_x / m_x \mathcal{F}_x}_{\substack{\text{viewed as} \\ \text{a skyscraper} \\ \text{sheaf}}} \longrightarrow 0$$

Some final ingredients are the following easy facts

Lemma 3 If X is a noetherian scheme, then the descending chain condition holds for closed sets:

Any chain of closed sets

$$X \supseteq X_1 \supseteq X_2 \cdots$$

stabilizes.

Lemma 4 If $\mathcal{F}$ is a coherent sheaf on a scheme X, the support

$$\operatorname{Supp}(\mathcal{F}) := \{ x \in X \mid \mathcal{F}_x \neq 0 \}$$

is closed.

Prop 2 For $n >> 0$, $\mathcal{F}(n)$ is globally generated.

pf By Nakayama's lemma, it's enough to show

$$H^0(\mathcal{F}(n)) \to H^0(\mathcal{F}_x(n)/m_x \mathcal{F}_x(n))$$

is surjective for all closed $x \in X$ and $n >> 0$.

Given a closed $x \in X$, we can choose n_1 s.t. for $n \geq n_1$, $H^1((m_x \cdot \mathcal{F})(n)) = 0$

This implies that

$$H^0(\mathcal{F}(n)) \to H^0(\mathcal{F}_x(n)/m_x \mathcal{F}(n))$$

is surjective. Equivalently that

$$x \notin X_1 = \bigcap_{n \geq n_1} \operatorname{Supp}\left(\mathcal{F}(n) / H^0(\mathcal{F}(n)) \otimes \mathcal{O}_{\mathbb{P}}\right)$$

If $X_1 \neq \emptyset$, we can repeat above argument to choose $x \in X_1 \setminus X_2$, where

$$X_2 = \bigcap_{n \geq n_2} \operatorname{Supp}\left(\mathcal{F}(n) / H^0(\mathcal{F}(n)) \otimes \mathcal{O}_{\mathbb{P}}\right)$$

for some $n_2 > n_1$

In this way, we get a chain

$$X \underset{\neq}{\supset} X_1 \underset{\neq}{\supset} X_2 \cdots$$

which must eventually stop at the emptyset //